Math 112C Final Exam practice problems

This preparation sheet contains problems from the second part of the course. The final exam will also contain (less than a half) problems from the first part. In order to prepare for those, use the Practice midterm sheet.

1. Consider the function \( f(z) = \frac{\sin z}{z^2(z-4i)} \). (a) Determine the poles and the order of each pole. (b) Evaluate the integral \( \oint_C f(z) \, dz \) where the contour \( C \) is a circle of radius 1.5 centered at \( z = i \). (c) Same as (b) but the circle is centered at point \( z = 2i \).

Solution:

(a) Recall that
\[
\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ..., 
\]
therefore
\[
f(z) = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ...}{z^2(z-4i)},
\]
or
\[
f(z) = \frac{1 - \frac{z^3}{3!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...}{z(z-4i)}. 
\]
Observe that the \( f(z) \) has an unremovable isolated singularity at \( z = 0 \), but the function \( zf(z) \) has a removable singularity at \( z = 0 \), therefore \( f(z) \) has a pole of order 1 at \( z = 0 \).

We see that there will be a possible pole at \( z = 4i \). Consider rewriting \( \sin z \) as
\[
\sin z = \sin((z-4i) + 4i) \\
= \sin(z-4i)\cos(4i) + \cos(z-4i)\sin(4i) \\
= \left((z-4i) - \frac{(z-4i)^3}{3!} + ...\right)\cos(4i) + \left(1 - \frac{(z-4i)^2}{2!}\right)\sin(4i) \\
= \sin(4i) + (z-4i)\cos(4i) + ...
\]
From this we see that \( f(z) \) has an unremovable singularity at \( z = 4i \), while \( (z-4i)f(z) \) has a removable singularity at \( z = 4i \), therefore \( f(z) \) has a pole of order 1 at \( z = 4i \).
(b) We note that the only pole in the contour $C$ is the one at $z = 0$, so we have

$$\int_C f(z) \, dz = 2\pi i \ \text{Res}_0 \left( \frac{\sin z}{z^2(z - 4i)} \right)$$

$$= 2\pi i \ \lim_{z \to 0} \frac{\sin z}{z(z - 4i)}$$

$$= 2\pi i \frac{1}{-4i}$$

$$= -\frac{\pi}{2}$$

(c) Since there are no singularities inside the contour, we have

$$\int_C f(z) \, dz = 0.$$
2. Calculate
\[ P.V. \int_{-\infty}^{\infty} \frac{\sin x}{x(1 + x^4)} \, dx. \]

Solution:

First we note that
\[ \sin x = \text{Im} \left( e^{ix} \right), \]
so we will calculate
\[ \int_{-\infty}^{\infty} \frac{e^{ix}}{x(1 + x^4)} \, dx \]
and take the imaginary part of the answer.

We consider a closed contour just as in problem 64.4 from the text. Then
\[
\int_C \frac{e^{iz}}{z(1 + z^4)} \, dz = \int_{-\epsilon}^{-L} \frac{e^{ix}}{x(1 + x^4)} \, dx + \int_{\Gamma_r} \frac{e^{iz}}{z(1 + z^4)} \, dz \\
+ \int_{\epsilon}^{L} \frac{e^{ix}}{x(1 + x^4)} \, dx + \int_{\Gamma_L} \frac{e^{iz}}{z(1 + z^4)} \, dz,
\]
from which we get
\[
\int_{-\infty}^{\infty} \frac{e^{ix}}{x(1 + x^4)} \, dx = \int_C \frac{e^{iz}}{z(1 + z^4)} \, dz - \lim_{\epsilon \to 0^+} \int_{\Gamma_\epsilon} \frac{e^{iz}}{z(1 + z^4)} \, dz - \lim_{L \to \infty} \int_{\Gamma_L} \frac{e^{iz}}{z(1 + z^4)} \, dz.
\]
First we consider integration on the larger semi-circle with radius \( L \). We know that
\[
\lim_{L \to \infty} \max_{z \in \Gamma_L} \left| \frac{1}{z(1 + z^4)} \right| = 0,
\]
so we have
\[
\lim_{L \to \infty} \int_{\Gamma_L} \frac{e^{iz}}{z(1 + z^4)} \, dz = 0
\]
by Jordan’s Lemma. (See Section 64 for a discussion on Jordan’s Lemma.)

Next we consider the contour integral on the smaller semi-circle with radius \( \epsilon \). We have
\[
\int_{\Gamma_\epsilon} \frac{e^{iz}}{z(1 + z^4)} \, dz = \int_{0}^{\pi} \frac{e^{i\epsilon e^{i\theta}}}{\epsilon e^{i\theta} (1 + \epsilon^4 e^{4i\theta})} i\epsilon e^{i\theta} \, d\theta \\
= -\int_{0}^{\pi} \frac{e^{-\epsilon (\sin \theta - i \cos \theta)}}{1 + \epsilon^4 e^{4i\theta}} \, d\theta.
\]
Now
\[
\lim_{\varepsilon \to 0^+} \frac{i e^{-\varepsilon \sin \theta - i \varepsilon \cos \theta}}{1 + e^4 \varepsilon^4 i \theta} = i,
\]
so
\[
\lim_{\varepsilon \to 0^+} \int_{\Gamma_\varepsilon} \frac{e^{iz}}{z(1 + z^4)} \, dz = -\pi i.
\]
We also need to find the value of the contour integral over the whole closed curve using residue theory. We note that the function \( e^{iz} \) has singularities at zero and wherever \( z^4 = -1 \). The singularity at zero will not be an issue since it is outside the closed contour. To find the other singularities we use the fact that
\[
z^4 = -1 = e^{k\pi i}
\]
where \( k \) is an odd integer. From this we see that the fourth roots of -1 are given by
\[
z = e^{\frac{k\pi i}{4}},
\]
where \( k = 1, 3, 5, 7 \). Since \( e^{\frac{5\pi i}{4}} \) and \( e^{\frac{7\pi i}{4}} \) are not in the closed contour, we need not calculate the residues for these points. Now we have
\[
\text{Res}_{e^{\frac{5\pi i}{4}}} \left( \frac{e^{iz}}{z(1 + z^4)} \right) = \left[ \frac{e^{iz}}{z - e^{\frac{5\pi i}{4}}} \right]_{z = e^{\frac{5\pi i}{4}}} = \frac{e^{\frac{5\pi i}{4}}}{(e^{\frac{3\pi i}{4}} - e^{\frac{5\pi i}{4}})(e^{\frac{3\pi i}{4}} - e^{\frac{7\pi i}{4}})}.
\]
This is equal to
\[
e^{-\sqrt{2}/2 + i\sqrt{2}/2 \pi} \frac{e^{\pi i} (1 - e^{\frac{3\pi i}{4}}) (1 - e^{\frac{5\pi i}{4}})}{-1 (1 - i) (1 - (-1)) (1 - (-i))}
\]
which simplifies to
\[
e^{-\sqrt{2}/2} \left(\cos(\sqrt{2}/2) + i \sin(\sqrt{2}/2)\right),
\]
and
\[4\]
Residue theory tells us that
\[
\int_C e^{iz} \frac{z}{z(1 + z^4)} dz = 2\pi i \left( \text{Res}_{e^{\frac{3\pi i}{4}}} \left( \frac{e^{iz}}{z(1 + z^4)} \right) + \text{Res}_{e^{\frac{3\pi i}{4}}} \left( \frac{e^{iz}}{z(1 + z^4)} \right) \right),
\]
so
\[
\int_C e^{iz} \frac{z}{z(1 + z^4)} dz = 2\pi i \cdot \frac{e^{-\sqrt{2}/2}}{4} \left[ (\cos(\sqrt{2}/2) + i\sin(\sqrt{2}/2)) + (\cos(\sqrt{2}/2) - i\sin(\sqrt{2}/2)) \right]
= -\pi i e^{-\sqrt{2}/2} \cos \left( \frac{\sqrt{2}}{2} \right).
\]
From all of the above we conclude that
\[
\int_{-\infty}^{\infty} \frac{e^{ix}}{x(1 + x^4)} dx = -\pi i e^{-\sqrt{2}/2} \cos \left( \frac{\sqrt{2}}{2} \right) + \pi i,
\]
therefore
\[
\int_{-\infty}^{\infty} \frac{\sin x}{x(1 + x^4)} dx = \pi \left[ 1 - e^{-\sqrt{2}/2} \cos \left( \frac{\sqrt{2}}{2} \right) \right].
\]
3. Find the solution of the heat equation:

\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad t > 0, \quad (1)
\]

\[
u(x, 0) = \begin{cases} 
1, & 0 \leq x \leq 3, \\
0, & \text{otherwise} \end{cases} \quad (2)
\]

Leave the answer in an integral form.

Solution:

To solve this problem we use Fourier transforms. Using the property

\[F[u_x] = -i\omega F[u],\]

we convert the PDE to

\[\hat{u}_t + \omega^2 \hat{u} = 0.\]

We must also convert the initial condition. We have

\[\hat{u}(\omega, 0) = \int_{-\infty}^{\infty} u(x, 0) e^{i\omega x} \, dx = \int_0^3 e^{i\omega x} \, dx = \frac{-i}{\omega} (e^{3i\omega} - 1).\]

The general solution to the ODE is

\[\hat{u}(\omega, t) = Ce^{-\omega^2 t},\]

and using the initial condition gives

\[\hat{u}(\omega, t) = \frac{i}{\omega} (1 - e^{3i\omega}) e^{-\omega^2 t}.\]

Now we convert back using the inverse Fourier transform:

\[u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i}{\omega} (1 - e^{3i\omega}) e^{-\omega^2 t} e^{-i\omega x} \, d\omega.\]
4. Solve
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad y_1 < y < y_2, \quad (3)
\]
\[
u(x, y_1) = \frac{1}{x^2 + 1}, \quad (4)
\]
\[
u(x, y_2) = 0. \quad (5)
\]
Leave the answer in an integral form.

Solution:

We will be needing the Fourier transform of \( u(x, y_1) = \frac{1}{x^2 + 1} \), which is found using the definition
\[
\hat{u}(\omega, y_1) = \int_{-\infty}^{\infty} e^{i\omega x} x^2 + 1 \, dx.
\]
Assume that \( \omega > 0 \). We consider a closed contour in the shape of a semi-circle in the upper half-plane as in the diagram on page 282 of the text. Then we have that
\[
\int_{C_R} e^{i\omega z} z^2 + 1 \, dz = \int_{-R}^{R} e^{i\omega x} x^2 + 1 \, dx + \int_{\Gamma_R} e^{i\omega z} z^2 + 1 \, dz.
\]
For the integral on the semi-circle, \( \Gamma_R \), we have
\[
\lim_{R \to \infty} \max_{z \in \Gamma_R} \left| \frac{1}{z^2 + 1} \right| = 0,
\]
therefore
\[
\lim_{R \to \infty} \int_{\Gamma_R} e^{i\omega z} z^2 + 1 \, dz = 0.
\]
As a result we know that
\[
\int_{-\infty}^{\infty} e^{i\omega x} x^2 + 1 \, dx = \int_{C_R} e^{i\omega z} z^2 + 1 \, dz,
\]
which can be calculated using residue theory. (See Theorem 1 on page 303.)
We have
\[
\int_{-\infty}^{\infty} e^{i\omega x} \frac{dx}{x^2 + 1} = 2\pi i \operatorname{Res}_{i} \left( \frac{e^{i\omega z}}{z^2 + 1} \right)
\]
\[
= 2\pi i \left[ \frac{e^{i\omega z}}{z + i} \right]_{z = i}
\]
\[
= 2\pi i e^{-\omega} - 2i
\]
\[
= \pi e^{-\omega}
\]
We must also calculate \( \hat{u}(\omega, y_1) \) for the case where \( \omega < 0 \). Here we consider a closed contour in the shape of a semi-circle in the bottom half-plane. Recall that the standard orientation is to integrate over the contour in a counter-clockwise direction. Then we have
\[
\int_{C_R} \frac{e^{i\omega z}}{z^2 + 1} \, dz = -\int_{-R}^{R} e^{i\omega x} \frac{dx}{x^2 + 1} + \int_{\Gamma_R} e^{i\omega z} \, dz.
\]
Again for the integral on the semi-circle, \( \Gamma_R \), we have
\[
\lim_{R \to \infty} \max_{z \in \Gamma_R} \left| \frac{1}{z^2 + 1} \right| = 0,
\]
which implies that
\[
\lim_{R \to \infty} \int_{\Gamma_R} \frac{e^{i\omega z}}{z^2 + 1} \, dz = 0.
\]
As a result we know that
\[
\int_{-\infty}^{\infty} e^{i\omega x} \frac{dx}{x^2 + 1} = -\int_{C_R} \frac{e^{i\omega z}}{z^2 + 1} \, dz,
\]
which can be calculated using residue theory. (See Theorem 2 on page 304.)
We have
\[
\int_{-\infty}^{\infty} e^{i\omega x} \frac{dx}{x^2 + 1} = -2\pi i \operatorname{Res}_{-i} \left( \frac{e^{i\omega z}}{z^2 + 1} \right)
\]
\[
= -2\pi i \left[ \frac{e^{i\omega z}}{z - i} \right]_{z = -i}
\]
\[
= -2\pi i e^{-\omega} - 2i
\]
\[
= \pi e^{-\omega}
\]
Therefore, we have that
\[ \hat{u}(\omega, y_1) = \pi e^{-|\omega|}. \]

Now we take the Fourier transform of the PDE. We have
\[ \begin{align*}
-\omega^2 \hat{u} + \hat{u}_{yy} &= 0, \\
\hat{u}(\omega, y_1) &= \pi e^{-|\omega|}, \\
\hat{u}(\omega, y_2) &= 0.
\end{align*} \]

The general solution to this ODE is
\[ \hat{u}(\omega, y) = c_1 e^{\omega y} + c_2 e^{-\omega y}. \]

The second boundary condition gives
\[ c_1 e^{\omega y_2} + c_2 e^{-\omega y_2} = 0, \]
so that
\[ c_2 = -c_1 e^{2\omega y_2}. \]

Then the solution has the form
\[ \hat{u}(\omega, y) = c \left( e^{\omega y} - e^{\omega(2y_2-y)} \right). \]

The first boundary condition gives
\[ c \left( e^{\omega y_1} - e^{\omega(2y_2-y_1)} \right) = \pi e^{-|\omega|}, \]
so
\[ c = \frac{\pi e^{-|\omega|}}{e^{\omega y_1} - e^{\omega(2y_2-y_1)}}. \]

The solution to the ODE for \( \hat{u}(\omega, y) \) is therefore
\[ \hat{u}(\omega, y) = \frac{\pi e^{-|\omega|}}{e^{\omega y_1} - e^{\omega(2y_2-y_1)}} \left( e^{\omega y} - e^{\omega(2y_2-y)} \right), \]
or
\[ \hat{u}(\omega, y) = \pi e^{-|\omega|} \frac{e^{-\omega(y_2-y)} - e^{\omega(y_2-y)}}{e^{-\omega(y_2-y_1)} - e^{\omega(y_2-y_1)}}. \]

Using the inverse Fourier transform we have
\[ u(x, y) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-|\omega|} \frac{e^{-\omega(y_2-y)} - e^{\omega(y_2-y)}}{e^{-\omega(y_2-y_1)} - e^{\omega(y_2-y_1)}} e^{-ix} \, d\omega. \]
5. Find the sine and cosine transforms of \( f(x) = e^{-5x} \) defined for \( x \geq 0 \).

**Solution:**

The sine transform is found as follows:

\[
F_s[f] = \int_0^\infty e^{-5x} \sin \omega x \, dx
\]

\[
= \lim_{L \to \infty} \int_0^L e^{-5x} \sin \omega x \, dx
\]

\[
= \lim_{L \to \infty} \left[ \frac{e^{-5x}}{\omega^2 + 25} (-5 \sin \omega x - \omega \cos \omega x) \right]_{x=0}^{x=L}
\]

\[
= \lim_{L \to \infty} \frac{\omega - e^{-5L} (5 \sin \omega L + \omega \cos \omega L)}{\omega^2 + 25}
\]

\[
= \frac{\omega}{\omega^2 + 25}
\]
The cosine transform is found as follows:

\[
F_{c}[f] = \int_{0}^{\infty} e^{-5x} \cos \omega x \, dx
\]

\[
= \lim_{L \to \infty} \int_{0}^{L} e^{-5x} \cos \omega x \, dx
\]

\[
= \lim_{L \to \infty} \left[ \frac{e^{-5x}}{\omega^2 + 25} (-5 \cos \omega x + \omega \sin \omega x) \right]_{x=0}^{x=L}
\]

\[
= \lim_{L \to \infty} \frac{5 - e^{-5L} (5 \cos \omega L - \omega \sin \omega L)}{\omega^2 + 25}
\]

\[
= \frac{5}{\omega^2 + 25}
\]
6. Solve the heat equation on a half-line:

\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad t > 0, \quad 0 < x < \infty, \quad (6)
\]

\[
u(x, 0) = e^{-2x}, \quad (7)
\]

\[
\nu(0, t) = 0. \quad (8)
\]

**Solution:**

Since \(u(0, t) = 0\), we seek a solution that is odd in \(x\). For this reason we use sine transforms to solve the problem. We require the sine transform of \(e^{-2x}\), which is given by

\[
F_s[e^{-2x}] = \frac{\omega}{\omega^2 + 4}.
\]

(This part is very similar to #5 above. Simply replace the 5 with a 2.) Then we convert the PDE using sine transforms. Recall from equation (68.8) on page 322 that

\[
F_s[u_{xx}] = \omega u(0, t) - \omega^2 F_s[u].
\]

So the transformed equation is

\[
\hat{u}_t + \omega^2 \hat{u} = 0,
\]

\[
\hat{u}(\omega, 0) = \frac{\omega}{\omega^2 + 4}.
\]

The general solution to the ODE is

\[
\hat{u}(\omega, t) = Ce^{-\omega^2 t},
\]

and the initial condition gives

\[
\hat{u}(\omega, t) = \frac{\omega}{\omega^2 + 4} e^{-\omega^2 t}.
\]

So using the inversion formula (68.3) on page 321, we have

\[
u(x, t) = \frac{2}{\pi} \int_0^\infty \frac{\omega}{\omega^2 + 4} e^{-\omega^2 t} \sin \omega x \, d\omega.
\]
7. Solve
\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + tu = 0, \quad t > 0, \quad 0 < x < \infty, \quad (9)
\]
\[
u(x, 0) = f(x), \quad (10)
\]
\[
\frac{\partial u}{\partial x}(0, t) = 0. \quad (11)
\]

**Solution:**

Since \(u_x(0,t) = 0\), we seek a solution that is even in \(x\). For this reason we use cosine transforms to solve the problem. Recall from equation (68.8) on page 322 that
\[
F_c[u_{xx}] = -u_x(0,t) - \omega^2 F_c[u].
\]
So the transformed equation is
\[
\hat{u}_t + \omega^2 \hat{u} + t\hat{u} = 0,
\]
\[
\hat{u}(\omega, 0) = \hat{f}(\omega).
\]
To solve the ODE, we use separation of variables. We have
\[
\frac{d\hat{u}}{dt} = - \left( \omega^2 + t \right) \hat{u}
\]
which leads to
\[
\int \frac{1}{\hat{u}} \, d\hat{u} = - \int \left( \omega^2 + t \right) \, dt,
\]
or
\[
\ln |\hat{u}| = -\omega^2 t - \frac{t^2}{2} + \tilde{C}.
\]
The general solution to the ODE is
\[
\hat{u}(\omega, t) = Ce^{-\omega^2 t - \frac{t^2}{2}},
\]
and the initial condition gives
\[
\hat{u}(\omega, t) = \hat{f}(\omega)e^{-\omega^2 t - \frac{t^2}{2}}.
\]
So using the inversion formula (68.6) on page 321, we have
\[
u(x, t) = \frac{2}{\pi} \int_0^{\infty} \hat{f}(\omega)e^{-\omega^2 t - \frac{t^2}{2}} \cos \omega x \, d\omega.
\]