Math 112A practice problems with solutions

ATTENTION: I entered answers to the problems after each problem. However, for the Midterm you will be required to show your calculations and explain the reasoning. An answer alone is not enough. Also, you will have to sketch graphs (which I didn’t do here for technical reasons).

1. Consider the wave equation on an infinite line, with $u(x, 0) = f(x)$ defined by

$$f(x) = \begin{cases} 
0, & x < 0, \\
x^2, & 0 \leq x \leq 1, \\
(2 - x), & 1 \leq x \leq 2, \\
0, & x > 2.
\end{cases}$$

Set $\partial u/\partial t(x, 0) = g(x) = 0$ and $c = 1/2$. Draw the solution at $t = 0$ and $t = 5$. Calculate the time, $t$, at which $u(15, t) = 1/2$.

**Solution:** $u(x, 0)$ is simply given by $f(x)$. $u(x, 5)$ is given by $1/2[f(x + 5/2) + f(x - 5/2)]$. This looks like 2 humps of height 1/2, each of them has the shape of the initial condition and the maxima are located at points $x = -3/2$ and $x = 7/2$. Finally, $u(15, t) = 1/2$ for $t = 28$.

2. Suppose that the string of problem 1 is finite, with boundaries at $x = 0$ and $x = 5$:

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{4} \frac{\partial^2 u}{\partial x^2} = 0,$$

$u(x, 0) = f(x)$, see Problem 1,

$$\frac{\partial u}{\partial t}(x, 0) = 0,$$

$u(0, t) = 0$,

$u(5, t) = 0$.

The initial condition has a cusp at $x = 1$. On an $x$–$t$ diagram (with $0 \leq t \leq 4$) show how this cusp will propagate. (*Hint: cusps propagate along characteristics.*)

**Solution:** Draw the $x$–$t$ diagram, with 2 characteristics. One of them has the equation $t = 2 - 2x$ for $0 \leq x \leq 1$ and then it is reflected from the left wall with the equation $t = 2 + 2x$, $0 \leq x \leq 1$. The other one is $t = -2 + 2x$ for $2 \leq x \leq 2$. The intersection of these characteristics with horizontal lines...
indicate the position of the cusp on the string at corresponding moments of time.

3. A string of length 3 is fixed at the right end, while the left end is a "sliding loop" (free). No external force is applied. Set $c=2$ and write down the initial-boundary value problem with the initial shape given by $x^2(3-x)$ and the initial velocity $x^2$. Draw the extension of the initial data outside the string. Also, find $u(1,4)$ and $u(3,10)$.

**Solution:** The equations are

\[
\frac{\partial^2 u}{\partial t^2} - 4 \frac{\partial^2 u}{\partial x^2} = 0,
\]
\[
u(x, 0) = x^2(3-x),
\]
\[
\frac{\partial u}{\partial t}(x, 0) = x^2,
\]
\[
\frac{\partial u}{\partial x}(0, t) = 0,
\]
\[
u(3, t) = 0,
\]

The extensions of both initial shape and initial velocity are drawn by an even reflection around $x = 0$ and an odd reflection around $x = 3$. $u(1, 4) = -10/3$. $u(3, 10) = 0$.

4. Consider a wave equation on an infinite line,

\[
\frac{\partial^2 u}{\partial t^2} - \frac{1}{x^4} \frac{\partial^2 u}{\partial x^2} = 0.
\]

Find the characteristics though the point $(0,3)$. Draw the domains of dependence and influence of the point $(0,3)$ (for $t \geq 0$).

**Solution:** The equations of characteristics are $t = \pm x^3/3 + 3$. The domain of dependence is below the two curves. The domain of influence is above.
5. Is the equation
\[
\frac{\partial^2 u}{\partial t^2} + 4 \frac{\partial^2 u}{\partial x \partial t} - \frac{9}{4} \frac{\partial^2 u}{\partial x^2} = 0
\]
hyperbolic, elliptic or parabolic (explain)? Find the general equations for characteristics if possible.

**Solution:** The equation is hyperbolic because \( A^2 - 4BC = 25 > 0 \). The characteristics are \( \xi = 2x + t, \eta = 2x - 9t \).

6. Given the initial boundary value problem, for \( 0 \leq x \leq 1 \),
\[
\frac{\partial^2 u}{\partial t^2} - 4 \frac{\partial^2 u}{\partial x^2} = 0,
\]
\( u(x,0) = x(1-x) \),
\( \frac{\partial u}{\partial t}(x,0) = \sqrt{x} \),
\( u(0,t) = t \),
\( u(1,t) = \sin t \),
write down the solution as \( u(x,t) = v(x,t) + w(x,t) \), where \( v(x,t) \) satisfies the boundary conditions and \( w(x,t) \) can be found as the D'Alembert solution of a problem with homogeneous boundary conditions. Find \( v(x,t) \) explicitly. Formulate the problem for \( w(x,t) \) (DO NOT SOLVE FOR \( w(x,t) \)).

**Solution:** \( v(x,t) = x \sin t + (1-x)t \). The problem for \( w \) is:
\[
\frac{\partial^2 w}{\partial t^2} - 4 \frac{\partial^2 w}{\partial x^2} = x \sin t,
\]
\( w(x,0) = x(1-x) \),
\( \frac{\partial w}{\partial t}(x,0) = \sqrt{x} - 1 \),
\( w(0,t) = 0 \),
\( w(1,t) = 0 \).