Practice Exam, Math 194

1. Suppose S is a set with 10 elements. How many subsets of S have an odd number of elements?

   Fix one element s of S, and let S' be the set obtained by removing s from S. There is a one-to-one correspondence between subsets of S' and subsets of S with an odd number of elements, defined as follows. If $T \subset S'$ has an odd number of elements, then we match it with $T \subset S$. If $T \subset S'$ has an even number of elements, then we match it with $T \cup \{s\} \subset S$. It is easy to check that this map is 1 to 1 and onto, so

   $$\text{the number of odd-order subsets of } S = \text{the number of subsets of } S' = 2^{|S'|} = 2^9.$$

2. For every positive integer k, let $f_1(k)$ denote the sum of the squares of the base 10 digits of k. For $n \geq 2$ let $f_n(k) = f_1(f_{n-1}(k))$. Find $f_{2011}(11)$.

   We compute the following table:

k	1	2	3	4	5	6	7	8	9	10	11	12
$f_k(11)$	2	4	16	37	58	89	145	42	20	4	16	37

   After this the values will repeat, with $f_{k+8}(11) = f_k(11)$ for $k \geq 2$. Since $2011 \equiv 3 \pmod 8$, we conclude that $f_{2011}(11) = f_3(11) = 16$.

3. Evaluate the sum

   $$\sum_{n=1}^{1{,}000{,}000} \frac{1}{\langle \sqrt{n} \rangle}$$

   where $\langle x \rangle$ denotes the integer *closest* to x.

   We have

   $$\langle \sqrt{n} \rangle = m \iff (m - 1/2)^2 < n \leq (m + 1/2)^2 \iff m^2 - m + 1 \leq n \leq m^2 + m$$

   (note that $(m \pm 1/2)^2$ is never an integer, $\lceil (m-1/2)^2 \rceil = m^2 - m + 1$, and $\lfloor (m+1/2)^2 \rfloor = m^2 + m$). Therefore for every $m \geq 1$, there are exactly $2m$ values of n such that $\langle \sqrt{n} \rangle = m$. The first n with $\langle \sqrt{n} \rangle = 1000$ is $1000^2 - 1000 + 1 = 999{,}001$, so

   $$\sum_{n=1}^{1{,}000{,}000} \frac{1}{\langle \sqrt{n} \rangle} = \sum_{m=1}^{999} \frac{2m}{m} + \frac{1000}{1000} = 1999$$

4. Show (without using a calculator or doing extensive computation) that

   $$\log_{2011} 2012 + \log_{2012} 2011 > 2$$

($\log_a b$ denotes the logarithm base a of b).

Writing ln *for natural log, we have*

$$\log_{2011} 2012 + \log_{2012} 2011 = \frac{\ln(2012)}{\ln(2011)} + \frac{\ln(2011)}{\ln(2012)}$$

For all positive real numbers $a \neq b$ we have $(a-b)^2 > 0$, so $a^2 + b^2 > 2ab$. Therefore

$$\frac{a}{b} + \frac{b}{a} = \frac{a^2+b^2}{ab} > 2.$$

Now apply this with $a = \ln(2012)$, $b = \ln(2011)$.

5. Let $a_1, a_2, \ldots, a_{65}$ be positive integers, none of which has a prime factor greater than 13. Prove that, for some i, j with $i \neq j$, the product $a_i a_j$ is a perfect square.

   Every a_i can be written uniquely in the form $2^{n_2} \cdot 3^{n_3} \cdot 5^{n_5} \cdot 7^{n_7} \cdot 11^{n_{11}} \cdot 13^{n_{13}}$. We associate to each a_i the "pigeonhole"

   $$(n_2 \pmod 2),\ n_3 \pmod 2),\ n_5 \pmod 2),\ n_7 \pmod 2),\ n_{11} \pmod 2),\ n_{13} \pmod 2))$$

   There are $2^6 = 64$ pigeonholes (2 possibilities for each of the 6 primes), and 65 a_i's, so by the pigeonhole principle there must be some a_i and a_j, $i \neq j$, in the same pigeonhole. Then when we factor $a_i a_j$ into primes, every exponent is even, so $a_i a_j$ is a square.

6. For every $n \geq 1$, let

   $$x_n = \sqrt{3 + \sqrt{3 + \sqrt{3 + \cdots \sqrt{3}}}}$$

   with n 3's. Show that $\lim_{n\to\infty} x_n$ exists, and find its value.

   Let $C = (1+\sqrt{13})/2$. Suppose the limit exists, and call it L. Then

   $$L^2 = (\lim x_n)^2 = \lim x_n^2 = \lim x_{n-1} + 3 = L + 3,$$

   so L is a root of the polynomial $x^2 - x - 3$, and $L \geq 0$, so the quadratic formula tells us that $L = (1+\sqrt{13})/2 = C$.

   We will show that the sequence of x_n's is increasing and bounded, so it converges. It will follow by the argument above that the limit is equal to C.

   The function $x^2 - x - 3$ is negative on the interval $[0, C)$, so for $x \in [0, C)$ we have

   $$x < \sqrt{3+x} < \sqrt{3+C} = C.$$

   Since $x_{n+1} = \sqrt{3 + x_n}$, and $x_1 = \sqrt{3} < C$, we see by induction that $x_n < x_{n+1} < C$ for every n. Therefore the sequence is increasing and bounded, so it converges, and $\lim_{n\to\infty} x_n = C$.