1. Find all polynomials $p(x)$ with real coefficients satisfying the differential equation
\[ 7 \frac{d}{dx} [xp(x)] = 3p(x) + 4p(x + 1), \quad -\infty < x < \infty. \]

Solution:
Suppose we have a solution of degree $n$, so that $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$. By looking at the coefficient of $x^n$, we have $7(n+1)a_n = 3a_n + 4a_n$, which implies $n = 0$ (in assuming that our polynomial has degree $n$, we have assumed $a_n \neq 0$). Then, we see that for any constant $c$, $p(x) = c$ satisfies our differential equation.

2. Show that
\[ 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n} \]
for all positive integers $n$.

Solution:
We proceed by induction on $n$. Notice $1 < 2\sqrt{1}$. Define
\[ f(n) = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}. \]
and assume as our induction hypothesis that $f(k) < 2\sqrt{k}$. Consider
\[ (*) \quad f(k+1) = f(k) + \frac{1}{\sqrt{k+1}} < 2\sqrt{k} + \frac{1}{\sqrt{k+1}}. \]
Since we would like an expression on the right involving $2\sqrt{k+1}$, it is natural to consider
\[ 2(\sqrt{k+1} - \sqrt{k}) = 2(\sqrt{k+1} - \sqrt{k})(\sqrt{k+1} + \sqrt{k}) = 2 \frac{1}{\sqrt{k+1} + \sqrt{k}} > \frac{1}{\sqrt{k+1}}. \]
Using this on the right-hand side of $(*)$, we have $f(k+1) < 2\sqrt{k+1}$.

3. Show that
\[ \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq 3 \]
for all positive real numbers $x$, $y$, and $z$.

Solution:
Using the Arithmetic Mean - Geometric Mean inequality, we have
\[ \frac{\frac{\frac{x}{y} + \frac{y}{z} + \frac{z}{x}}{3}}{3} \geq \sqrt[3]{\frac{x}{y} \cdot \frac{y}{z} \cdot \frac{z}{x}} = 1, \]
and our result follows.
4. Let $T$ be an acute triangle. Inscribe a pair of rectangles $R$ and $S$ in $T$ as shown in the figure below. Let $A(X)$ denote the area of any polygon $X$. Find the maximum value of $\frac{A(R) + A(S)}{A(T)}$, where $T$ ranges over all acute triangles, and $R$ and $S$ range over all inscribed rectangles.

Solution:

Let $b_R, h_R, b_S, h_S, b_T,$ and $h_T$ be the lengths of the bases and heights of $R, S,$ and $T$, respectively. Observing the similarity of three triangles, we have

$$\frac{b_T}{h_T} = \frac{b_R}{h_T - h_R} = \frac{b_S}{h_T - h_R - h_S}.$$ 

Thus

$$A(S) = b_S h_S = \frac{b_T}{h_T} (h_T - h_R - h_S) h_S,$$

and for any fixed $R$ and $T$, this area is maximized when $h_S = \frac{1}{2} (h_T - h_R)$ (see this using properties of parabolas or the first derivative test). Also,

$$A(R) = b_R h_R = \frac{b_T}{h_T} (h_T - h_R) h_R.$$ 

Thus, for any fixed $R$ and $T$, if we choose $A(S)$ to be maximal,

$$\frac{A(R) + A(S)}{A(T)} = \frac{\frac{b_T}{h_T} (h_T - h_R) h_R + \frac{b_T}{h_T} (\frac{b_T}{h_T} - h_R)^2}{\frac{1}{2} b_T h_T}.$$ 

For any fixed $T$, this quantity is maximized when $h_R = \frac{1}{3} h_T$ (again using properties of parabolas or the first derivative test). Substituting this in for $h_R$, we see that the maximum value of $\frac{A(R) + A(S)}{A(T)}$ is $\frac{2}{3}$, independent of $T$.

Is there a better solution?
5. Let $a_1, a_2, \ldots, a_{100}$ be integers. Show that there exist $i, j, k,$ and $l$ with $i \neq j$ and $i \neq l$ such that $a_i - a_j + a_k - a_l$ is a multiple of 2004.

Solution:
Consider the multiset $S = \{a_n + a_m | 1 \leq n < m \leq 100\}$. This multiset has $\binom{100}{2} = 4950 > 2004$ elements, and so by the pigeonhole principle, two of these elements must be congruent modulo 2004. Let these two elements be $a_c + a_d$ and $a_j + a_l$. Since $\{c, d\} \neq \{j, l\}$, one element of $\{c, d\}$ is $\notin \{j, l\}$; set $i$ to be that element of $\{c, d\}$, and $k$ to be the other. Then $a_i - a_j + a_k - a_l$ is a multiple of 2004, with $i \neq j$ and $i \neq l$.

6. Find all real valued functions $F(x)$ defined for all real $x \neq 0, 1$ satisfying the functional equation

$$F(x) + F\left(\frac{x-1}{x}\right) = 1 + x.$$ 

Solution:
Notice that for any $x \neq 0, 1$,

$$F(x) + F\left(\frac{x-1}{x}\right) = 1 + x,$$

$$F\left(\frac{x-1}{x}\right) + F\left(\frac{1}{1-x}\right) = 1 + \frac{x-1}{x}, \text{ and}$$

$$F\left(\frac{1}{1-x}\right) + F(x) = 1 + \frac{1}{1-x}.$$ 

Thus

$$2F(x) = F(x) + F\left(\frac{x-1}{x}\right) - F\left(\frac{x-1}{x}\right) - F\left(\frac{1}{1-x}\right) + F\left(\frac{1}{1-x}\right) + F(x)$$

$$= 1 + x - \left(1 + \frac{x-1}{x}\right) + 1 + \frac{1}{1-x}$$

$$= 1 + x + \frac{1-x}{x} + \frac{1}{1-x},$$

and

$$F(x) = \frac{1}{2} \left(1 + x + \frac{1-x}{x} + \frac{1}{1-x}\right).$$