1. Show that some multiple of 1232123432123454321 contains all 10 digits (at least once) when written in base 10.

Solution 1: Let \( n = 1232123432123454321 \). Consider the \( n \) consecutive integers \( 1234567890 \times 10^{20}, 1234567890 \times 10^{20} + 1, \ldots, 1234567890 \times 10^{20} + n - 1 \). Each of these integers contains all 10 digits, and since there are \( n \) consecutive integers, one of them must be a multiple of \( n \).

Solution 2: Let \( n = 1232123432123454321 \). Notice that for any integer \( 0 \leq j \leq 9 \), \( jn \) has last digit \( j \). Thus, starting with the last digit and counting to the left, taking every 100th digit of

\[
9n \times 10^{900} + 8n \times 10^{800} + \cdots + 1n \times 10^{100} + 0n \times 10^0,
\]

we get all 10 digits.

2. For each integer \( n \geq 0 \), let \( S(n) = n - m^2 \), where \( m \) is the greatest integer with \( m^2 \leq n \). Define a sequence \( (a_k)_{k=0}^{\infty} \) by \( a_0 = A \) and \( a_{k+1} = a_k + S(a_k) \) for \( k \geq 0 \). For what positive integers \( A \) is this sequence eventually constant? (Putnam, 1991)

Solution: This sequence is eventually constant if and only if \( A \) is a square. If some \( a_k \) is a square, then \( S(a_k) = 0 \), so that \( a_{k+1} = a_k + S(a_k) = a_k \), and the sequence is constant from that point on. In particular, if \( a_0 = A \) is a square, then the sequence is constant. If \( a_k \) is not a square, then \( a_{k+1} = a_k + S(a_k) > a_k \), so the sequence is not constant from that point on. Moreover \( a_{k+1} = a_k + S(a_k) = S(a_k) + m^2 + S(a_k) = m^2 + 2S(a_k) \), where \( m \) is the greatest integer with \( m^2 \leq a_k \). Now notice that \( a_{k+1} > m^2 \), and \( a_{k+1} \neq (m+1)^2 \) because \( a_{k+1} \) has the same parity as \( m^2 \). Also, since \( S(a_k) < (m+1)^2 - m^2 < (m + 2)^2 - (m + 1)^2 \), \( a_{k+1} < m^2 + [(m + 1)^2 - m^2] + [(m + 2)^2 - (m + 1)^2] = (m + 2)^2 \). Since \( m^2 < a_{k+1} < (m+2)^2 \), and \( a_{k+1} \neq (m+1)^2 \), \( a_{k+1} \) is not a square. Thus, if \( a_0 = A \) is not a square, no \( a_k \) is a square, and so the sequence is not constant from any point on.

3. Determine \( F(x) \) if, for all real \( x \) and \( y \), \( F(x)F(y) - F(xy) = x + y \).

Solution: Notice that \( F(0)F(0) - F(0 \cdot 0) = 0 \), so \( F(0)[F(0) - 1] = 0 \), which implies that \( F(0) = 0 \) or 1. Also notice that for all \( x \), \( F(x)F(0) - F(x \cdot 0) = [F(x) - 1]F(0) = x \), so \( F(0) \neq 0 \), and thus \( F(0) = 1 \). Therefore \( [F(x) - 1] \cdot 1 = x \), and we have that \( F(x) = x + 1 \) for all \( x \). Finally, observe that \( F(x)F(y) - F(xy) = (x + 1)(y + 1) - (xy + 1) = xy + x + y + 1 - xy - 1 = x + y \) does hold for all real \( x \) and \( y \).
4. A two-person game is played as follows. The players alternate placing a penny on a circular table. Each penny must lie completely on the table, and not overlap any previously-placed pennies. The first player unable to fit a penny on the table loses. (You can assume they have all the pennies they need.) Is it better to go first or second? Is there a winning strategy? Is the answer any different if the table is square? triangular?

Solution: If a circular or square table fits at least one penny, it is better to go first. The player who plays first can force a win by placing the first penny in the exact center of the table. After each subsequent move by the second player, the first player can respond by playing in the spot that is the same distance from the center of the table, but rotated 180°. By the rotational symmetry of the table, the first player will always be able respond to any move by the second player, and therefore the first player will win.

5. Compute the determinant of the \( n \times n \) matrix all of whose diagonal entries are 0, and all of whose off-diagonal entries are 1.

Solution 1: Let \( A \) be the matrix described above. Since determinants are invariant under the row operation of adding one row to another, \( \det A = \det B \), where \( B \) is the matrix we get by replacing the bottom row of \( A \) by the sum of all the rows of \( A \). Also \( \det B = (n - 1) \det C \), where \( C \) is the matrix we get by dividing all of the entries in the bottom row of \( B \) by \( n - 1 \). Further, \( \det C = \det D \), where \( D \) is the matrix we get by subtracting the bottom row of \( C \) from each of its other rows, and then replacing the bottom row of that matrix by the sum of all its rows. The matrix \( D \) will then be the identity matrix with all but the last of its 1s replaced by \(-1\)s, and so \( \det A = (n - 1) \det D = (n - 1)(-1)^{n-1} \).

Solution 2: Let \( A \) be the matrix described above. Observe that \( n - 1 \) is the eigenvalue associated to the vector that is 1 in every component. Also notice that since \( \det(A + I_n) = 0 \), \(-1\) is a a solution to \( \det(A - \lambda I_n) = 0 \), and thus \(-1\) is also an eigenvalue of \( A \). Since the rank of \( (A + I_n) \) is one, the dimension of the eigenspace associated to \( \lambda = -1 \) is \( n - 1 \), and so the eigenvalue \(-1\) has multiplicity \( n - 1 \). The determinant of \( A \) is the product of its eigenvalues, so \( \det A = (n - 1)(-1)^{n-1} \).

6. Show that every infinite sequence of distinct real numbers contains either a strictly increasing subsequence or a strictly decreasing subsequence.

Solution: Assume that a sequence \((x_n)_{n=1}^\infty\) does not have a strictly increasing subsequence. Then for each \( N \), the maximum of \( \{x_n : n > N\} \) must exist. For each \( N \), let \( m(N) \) be that maximum, and let \( k_N \) be such that \( x_{k_N} = m(N) \). Let \( y_1 = m(1) \), and for \( i \geq 1 \), let \( y_{i+1} = m(k_i) \). Since \( m(N) \) is a weakly decreasing function and the \( x_n \) are distinct, we have \( y_{i+1} < y_i \) for all \( i \geq 1 \), and so \( (y_i)_{i=1}^\infty \) is a strictly decreasing subsequence of \((x_n)_{n=1}^\infty\).
7. (a) Give a sensible definition of the infinite tower of exponentials \( t(x) := x^{x^{x^{x^{ \ldots}}}} \) for real numbers \( x \geq 1 \), when it makes sense.

(b) Show that \( t(\sqrt{2}) = 2 \).

(c) Show that there is no real number \( a \) such that \( t(a) = 4 \).

(d) What can you say about the domain and range of \( t \)?

Solution:

(a) For \( x \geq 1 \), let \( a_0 = a_0(x) = x \), and for all \( n \geq 0 \), let \( a_{n+1} = a_{n+1}(x) = x^{a_n} \). Define \( t(x) = \lim_{n \to \infty} a_n \) if that limit exists.

(b) First notice that \( a_0(\sqrt{2}) = \sqrt{2} < \sqrt{\sqrt{2}^2} = a_1(\sqrt{2}) \). Also notice that for \( k \geq 0 \), \( a_k(\sqrt{2}) > a_{k-1}(\sqrt{2}) \) implies \( a_{k+1}(\sqrt{2}) = \sqrt{2}^{a_k(\sqrt{2})} > \sqrt{2}^{a_{k-1}(\sqrt{2})} = a_k(\sqrt{2}) \), so the sequence \( (a_n)_{n=0}^\infty \) is monotone increasing by induction. Next notice that if \( a_k(\sqrt{2}) < 2 \), then \( a_{k+1}(\sqrt{2}) = \sqrt{2}^{a_k(\sqrt{2})} < \sqrt{2}^2 = 2 \). Thus \( a_n(\sqrt{2}) < 2 \) for all \( n \). Since \( (a_n)_{n=0}^\infty \) is monotone increasing and bounded above, it must converge. Let \( t(\sqrt{2}) = L \). Then \( L = \lim_{n \to \infty} a_n(\sqrt{2}) = \lim_{n \to \infty} \sqrt{2}^{a_{n-1}(\sqrt{2})} = \sqrt{2}^L \). Solving for \( L \), since we have shown that \( L \leq 2 \), we may conclude that \( L = 2 \).

(c) Suppose for some \( a, t(a) = 4 \). Then \( 4 = \lim_{n \to \infty} a_n(a) = \lim_{n \to \infty} a^{a_{n-1}(a)} = a^4 \), which, since \( t(x) \) is defined for real numbers \( x \geq 1 \), would imply that \( a = \sqrt{2} \). However, in part (b) we have shown that \( t(\sqrt{2}) = 2 \).

(d) When \( t(x) \) exists, we must have \( t(x) = x^{t(x)} \), which implies that \( \ln(t(x)) = t(x) \ln(x) \). However, the equation \( \ln(y) = y \cdot c \) only has solutions when \( c \leq 1/e \), and so for \( t(x) \) to exist, we must have \( 1 \leq x \leq e^{1/e} \). For values of \( x \) in that range, let \( x = z^{1/z} \) for some \( 1 \leq z \leq e \). By the argument given in part (b), we can show that \( (a_n(x))_{n=0}^\infty \) is monotone increasing, bounded above by \( z \), and we must have that \( t(x) = z \). Thus the domain of \( t(x) \) is \([1, e^{1/e}]\), and the range is \([1, e] \).

8. Suppose \( x_0 \) and \( x_1 \) are given real numbers, and for \( n \geq 2 \) define

\[
x_n = \frac{x_{n-1} + x_{n-2}}{2}.
\]

Find \( \lim_{n \to \infty} x_n \).

Solution: Notice that \( x_n - x_{n-1} = -\frac{x_{n-1} - x_{n-2}}{2} \). Iterating, we have \( x_n - x_{n-1} = (-1)^{n-1}\frac{x_1 - x_0}{2^{n-1}} \). Thus \( x_n = x_0 + \sum_{i=1}^{n} (x_i - x_{i-1}) = x_0 + \sum_{i=1}^{n} (-1)^{i-1}\frac{x_1 - x_0}{2^{i-1}} = x_0 + (x_1 - x_0)\sum_{i=1}^{n} (-1)^{i-1}\frac{1}{2^{i-1}} \). Using the formula for the sum of a geometric series, we have that

\[
\lim_{n \to \infty} x_n = x_0 + (x_1 - x_0) \frac{1}{1 - (-\frac{1}{2})} = x_0 + 2x_1.
\]