Math 194, solutions for problem set #6

(1) The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find $c$ so that the areas of the two shaded regions are equal. (Putnam, 1993)

Solution: Let the final intersection of $y = c$ and $y = 2x - 3x^3$ occur at $x = d$. We then have $\int_0^d 2x - 3x^3 - c = 0$ so that $d^2 - \frac{3}{4}d^4 - cd = 0$. Solving for $c$, we have $c = d - \frac{3}{4}d^3$, and by the definition of $d$, we have $c = 2d - 3d^3$. Thus $0 = d - \frac{9}{4}d^3 = d(1 - \frac{3}{2}d^2)(1 + \frac{3}{2}d)$, so our $d$ must be $\frac{2}{3}$. Thus $c = 2(\frac{2}{3}) - 3(\frac{2}{3})^3 = \frac{4}{9}$.

(2) A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval $(a, b)$ and a nonzero function $g$ defined on $(a, b)$ such that this wrong product rule is true for $x$ in $(a, b)$.

(Putnam 1988)

Solution: Suppose $(fg)' = f'g + fg' = f'g'$. Then $g' = g + \frac{f}{f'}g' = g + \frac{1}{2x}g'$. Solving for $g'$, we have $g' = \frac{2x}{2x-1}g$. Solving this differential equation, we have

$$g = c \cdot e^{\int \frac{2x}{2x-1} dx} = c \cdot e^{x + \frac{1}{2} \ln(2x-1)} = c \cdot e^x \sqrt{2x-1}$$

for any constant $c$. To justify all of our operations, we must have $x > 1/2$, so for any interval $(a, b) \subset (1/2, \infty)$ and for any nonzero $c$, the wrong product rule holds for $f$ and $g$ as defined above.

(3) If $n$ is a positive integer, prove for $x > 0$ that $\frac{x^n}{(x + 1)^{n+1}} \leq \frac{n^n}{(n + 1)^{n+1}}$.

Solution: Let $f_n(x) = \frac{x^n}{(x + 1)^n+n+1}$, so that

$$f_n'(x) = \frac{nx^{n-1}(x + 1)^{n+1} - (n + 1)(x + 1)^n x^n}{(x + 1)^{2(n+1)}}$$

$$= \frac{x^{n-1}(x + 1)^n[n(x + 1) - (n + 1)x]}{(x + 1)^{2(n+1)}} = \frac{x^{n-1}(x + 1)^n(n - x)}{(x + 1)^{2(n+1)}}.$$  

From this, we see that $f_n'(x) > 0$ for $x \in (0, n)$ and $f_n'(x) < 0$ for $x \in (n, \infty)$, so $f_n(x)$ is maximized at $x = n$. 
(4) (a) Assuming that temperature is a continuous function, show that at any given
time on the earth’s equator there are two directly opposite points that have the
same temperature.

(b) A rock climber starts to climb a mountain at 7:00 AM on Saturday and gets
to the top at 5:00 PM. She camps on top and climbs back down on Sunday,
starting at 7:00 AM. Show that at some time of day on Sunday she was at the
same elevation as she was at that time on Saturday.

Solution to (a): Let \( t(x) \) be the difference in temperature between the point at \( x \)
degrees west of the prime meridian on the equator and the point directly opposite it
on Earth. Since \( t(0) = -t(180) \), either \( t(0) = 0 \), and we are done, or \( t(0) \) and \( t(180) \)
are of opposite sign. In the latter case, the intermediate value theorem insures us
that for some \( y \in (0, 180) \), \( t(y) = 0 \).

Solution to (b): Let \( f(t) \) be the elevation of the rock climber \( t \) hours after 7:00 AM
on Saturday, and let \( g(t) \) be the elevation of the rock climber \( t \) hours after 7:00
AM on Sunday. Since \( f \) and \( g \) are continuous, \( f - g \) is continuous. Also, since
\( (f - g)(0) < 0 \) and \( (f - g)(10) > 0 \), the intermediate value theorem insures us that
for some \( s \in (0, 10) \), \( (f - g)(s) = 0 \).

(5) Suppose \( f \) and \( g \) are differentiable functions and for every \( x \), \( f'(x)g(x) \neq f(x)g'(x) \).
Show that between every two zeros of \( f \) there is a zero of \( g \).

Solution: Let \( a \) and \( b \) be zeros of \( f \) with \( a < b \), and suppose \( g(x) \neq 0 \) for any
\( x \in (a, b) \). Let \( h(x) = \frac{f(x)}{g(x)} \). Since \( g(a) \) and \( g(b) \) are both nonzero by our hypothesis,
\( h(a) = h(b) = 0 \), and \( h(x) \) is differentiable on \( (a, b) \). By Rolle’s Theorem, there exists
some \( c \in (a, b) \) such that \( h'(c) = 0 \). However,

\[
h'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g^2(c)} \neq 0
\]

by our hypothesis, a contradiction. Thus we must have \( g(x) = 0 \) for some \( x \in (a, b) \).

(6) (a) Suppose that \( f(x) \) is continuous and \( f(x) \geq 0 \) on \([0, 1]\). Show that if
\( \int_0^1 (x - 1)^2 f(x) dx = 0 \), then \( f(x) = 0 \) on \([0, 1]\).

(b) Find all continuous, positive functions \( f(x) \), \( 0 \leq x \leq 1 \) such that

\[
\int_0^1 f(x) dx = 1, \quad \int_0^1 x f(x) dx = \alpha, \quad \int_0^1 x^2 f(x) dx = \alpha^2
\]

where \( \alpha \) is a given real number. \hspace{1cm} \text{(Putnam, 1964)}

Solution to (a): Suppose for some \( a \in [0, 1] \), \( f(a) \neq 0 \). Since \( f \) is continuous, there
exists an interval \( I \subset [0, 1] \) containing \( a \) such that \( f(x) > f(a)/2 \) for all \( x \in I \). Let
\( [b, c] \) be the middle third of \( I \). Then, since \((x - 1)^2\) is positive and \( f(x) \geq 0 \) on \((0, 1)\),

\[
\int_0^1 (x - 1)^2 f(x) dx \geq \int_b^c (x - 1)^2 f(x) dx > (c - 1)^2 \frac{f(a)}{2} (c - b) > 0.
\]

Thus, if \( \int_0^1 (x - 1)^2 f(x) dx = 0 \), we must have \( f(x) = 0 \) for all \( x \in [0, 1] \).
Solution to (b): Since \((x - \alpha)^2\) and \(f(x)\) are positive unless \(x = \alpha\), by the argument in part (a) (choosing \([b, c]\) to not contain \(\alpha\)), we have \(\int_0^1 (x - \alpha)^2 f(x)dx > 0\). However, \(\int_0^1 (x - \alpha)^2 f(x)dx = \int_0^1 \left[ x^2 f(x) - 2\alpha x f(x) + \alpha^2 f(x) \right] dx = \alpha^2 - 2\alpha^2 + \alpha^2 = 0\). Thus there is no continuous, positive function \(f(x)\) that meets the desired criteria.

(7) Suppose \(f\) is a differentiable function on \([0, 1]\), \(f(0) = 0\), and \(f'(x)\) is strictly increasing. Show that \(f(x)/x\) is strictly increasing.

Solution: For any fixed \(x_0 \in [0, 1]\), consider \(g(x) = f(x) - x\frac{f(x)}{x_0}\). By our hypotheses, \(g\) is differentiable, and \(g(0) = 0 = g(x_0)\). By Rolle’s Theorem, there exists \(c \in (0, x_0)\) such that \(g'(c) = 0\). Since \(f'(x)\) is strictly increasing, \(x_0 f'(x_0) > x_0 f'(c) = x_0 g'(c) + x_0 \frac{f(x_0)}{x_0} = f(x_0)\). Thus \(\left(\frac{f(x)}{x}\right)' \bigg|_{x=x_0} = \frac{x_0 f'(x_0) - f(x_0)}{x_0^2}\), and so \(f(x)/x\) is strictly increasing on \([0, 1]\).

(8) Suppose \(f\) is a continuous function on \([0, 1]\), \(n \in \mathbb{Z}^+, \int_0^1 x^k f(x)dx = 0\) for \(k = 0, 1, \ldots, n - 1\), and \(\int_0^1 x^n f(x)dx = 1\). Show that there is a \(c \in [0, 1]\) such that \(|f(c)| > 2^n(n+1)|\).

Solution: Applying the Binomial Theorem,
\[
\int_0^1 \left(x - \frac{1}{2}\right)^n f(x)dx = \int_0^1 \sum_{k=0}^n \binom{n}{k} x^k \left(-\frac{1}{2}\right)^{n-k} f(x)dx = \int_0^1 x^n f(x)dx = 1,
\]
since \(\int_0^1 x^k f(x)dx = 0\) for \(k = 0, 1, \ldots, n - 1\). Letting \(M = \sup\{|f(x)| : 0 \leq x \leq 1\}\), we also have
\[
\int_0^1 \left(x - \frac{1}{2}\right)^n f(x)dx \leq \int_0^1 \left|\left(x - \frac{1}{2}\right)^n\right| \cdot |f(x)|dx < M \int_0^1 \left|\left(x - \frac{1}{2}\right)^n\right| dx
= 2M \int_{\frac{1}{2}}^1 \left(x - \frac{1}{2}\right)^n dx = 2M \frac{\left(\frac{1}{2}\right)^{n+1}}{n + 1},
\]
where the strict inequality follows from the fact that \(|f(x)|\) cannot be constant, since \(\int_0^1 f(x)dx = 0\) while \(\int_0^1 x^n f(x)dx = 1\). Combining these, we have \(M > 2^n(n+1)\).