(1) Let $0 < x_i < \pi$, $i = 1, \ldots, n$ and set $x = (x_1 + \cdots + x_n)/n$. Prove that
\[
\prod_{i=1}^{n} \left( \frac{\sin x_i}{x_i} \right) \leq \left( \frac{\sin x}{x} \right)^n.
\] (Putnam, 1978)

(2) If $a, b, c$ are positive real numbers, show that
\[
\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.
\]

(3) For every positive integer $n$, show that
\[
\sqrt{1 + \sqrt{2 + \sqrt{3 + \cdots + \sqrt{n}}}} < 2.
\]

(4) If $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ is an injective function, then for every $n$
\[
\sum_{k=1}^{n} \frac{f(k)}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}.
\]

(5) Prove that for every positive integer $n$
\[
\frac{n^n}{e^{n-1}} \leq n! \leq \frac{n^{n+1}}{e^{n-1}}.
\]

(6) Let $f(x)$ be a function such that $f(1) = 1$ and for $x \geq 1$
\[
f'(x) = \frac{1}{x^2 + f(x)^2}.
\]
Prove that $\lim_{x \to \infty} f(x)$ exists and is less than $1 + \frac{\pi}{4}$. (Putnam, 1947)

(7) Show that if $\epsilon(n) = 1/n$, then for every $n \geq 1$,
\[
2\sqrt{n^2 + n} - 2n - \epsilon(n) < \sum_{i=1}^{n} \frac{1}{\sqrt{n^2 + i}} < 2\sqrt{n^2 + n} - 2n.
\]
Can you improve the “error term” $\epsilon(n)$?