Math 230C, problem set #2

due April 20

(1) Suppose $\rho : G \to \mathrm{GL}_n(k)$ is a representation. Show that the map $g \mapsto \det(\rho(g))$ is a 1-dimensional representation of G.

(2) Suppose $\rho : G \to \mathrm{GL}_n(\mathbf{C})$ is a representation. Show that for every $g \in G$, $\rho(g)$ is diagonalizable.

(3) Suppose V is a representation of the group G, and let χ denote its character.
  (a) What is $\chi(1)$?
  (b) Prove that $|\chi(g)| \leq |G|$ for every $g \in G$.
  (c) Suppose that the representation is faithful and $|\chi(g)| = |G|$. What can you conclude?

(4) Suppose that $V = k^n$ and χ is the character of the representation of the symmetric group S_n on V by permuting the vectors in a basis. If $\sigma \in S_n$, what is $\chi(\sigma)$?

(5) Suppose that χ is the character of a representation on G on a complex vector space V. Let $W = \{v \in V : gv = v \text{ for every } g \in G\} \subset V$. Show that W is a subspace of V and

$$\dim_{\mathbf{C}} W = \frac{1}{|G|} \sum_{g \in G} \chi(g).$$

(6) Describe explicitly the character of the natural representation of the dihedral group D_{2n} on $\mathbf{R}^2$.

(7) Suppose that $\rho : G \to \mathrm{GL}_n(\mathbf{C})$ is a representation and χ is its character. If $\rho = \oplus_{i=1}^n \rho_i$ with irreducible ρ_i, prove that

$$n = \langle \chi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2.$$

(8) Let ρ be the 4-dimensional representation of the quaternion group Q_8 acting on the Hamilton quaternions $\mathbb{H}$ by left multiplication.
  (a) Write down the matrices $\rho(g)$ for each $g \in G$.
  (b) Write down $\chi(g)$ for each $g \in G$.
  (c) Prove that $\rho : Q_8 \to \mathrm{GL}_2(\mathbf{R})$ is irreducible.
  (d) Prove that $\rho : Q_8 \to \mathrm{GL}_2(\mathbf{C})$ is reducible.