Math 124, problem set #1

Z, **Q**, and **R** denote the sets of integers, rational numbers, and real numbers, respectively.

(1) Which of the following are fields? Justify your answer.
   (a) $\{a + b\sqrt{2} : a, b \in \mathbf{Z}\}$
   (b) $\{a + b\sqrt{1/2} : a, b \in \mathbf{Q}\}$
   (c) $\{a + b\sqrt{\pi} : a, b \in \mathbf{Q}\}$
   (d) $\{a + b\sqrt{\pi} : a, b \in \mathbf{R}\}$
   (e) $\{a\sqrt{2} + b\sqrt{3} : a, b \in \mathbf{Q}\}$

(2) Show that $\{a + b\sqrt[3]{2} + c\sqrt[3]{4} : a, b, c \in \mathbf{Q}\}$ is a field. You may use the fact that

$$(a + b\sqrt[3]{2} + c\sqrt[3]{4})(a^2 - 2bc + (-ab + 2c^2)\sqrt[3]{2} + (b^2 - ac)\sqrt[3]{4}) = a^3 + 2b^3 - 6abc + 4c^3$$

(3) Suppose that $z \in \mathbf{R}$ and $\mathbf{F} = \{a + bz : a, b \in \mathbf{Q}\}$ is a field.
   (a) Show that $z = r + s\sqrt{t}$ for some rational numbers r, s, and t.
   (b) Conclude that $\mathbf{F} = \mathbf{Q}(\sqrt{t})$ for some rational number t.

(4) Describe how you would construct a segment of length

$$\sqrt{7 + \sqrt{3 + \sqrt{2}} - \sqrt{5}} - \frac{1 + \sqrt{2}}{\sqrt{3}}$$

What is the sequence of field extensions (as in Theorem 1, p. 21, from the textbook) used in your construction?

(5) If $\mathbf{F}$ is a field and $\varphi : \mathbf{F} \to \mathbf{F}$ is a function, we say that φ is a homomorphism if for every $x, y \in F$, we have $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(xy) = \varphi(x)\varphi(y)$.
   Let $f : \mathbf{Q}(\sqrt{2}) \to \mathbf{Q}(\sqrt{2})$ be the function defined by $f(a + b\sqrt{2}) = a - b\sqrt{2}$ if $a, b \in \mathbf{Q}$.
   (a) Show that f is well-defined (that is, show that there is only one way to write an element of $\mathbf{Q}(\sqrt{2})$ as $a + b\sqrt{2}$ with $a, b \in \mathbf{Q}$, so the definition of f is unambiguous).
   (b) Show that f is a homomorphism.
   (c) Show that if $g : \mathbf{Q}(\sqrt{2}) \to \mathbf{Q}(\sqrt{2})$ is a homomorphism, then either $g = f$ or g is the identity function. (Hint: what is $g(2)$? What is $g(\sqrt{2})$?)