Math 124, problem set #2

due April 18, 2006

As usual, $\mathbf{Q}$ is the field of rational numbers.

(1) Using the fact that $\sqrt[3]{2}$ is not constructible, give a simple proof that $\sqrt[3]{4}$ is not constructible.

(2) Suppose $\mathbf{F}$ is a field, $k \in \mathbf{F}$, and $\sqrt{k} \notin F$.

(a) Show that the function $f : \mathbf{F}(\sqrt{k}) \to \mathbf{F}(\sqrt{k})$ defined by

$$f(a + b\sqrt{k}) = a - b\sqrt{k}$$

is a homomorphism (see problem 5 from problem set 1).

(b) Show that if $p(x)$ is a polynomial with coefficients in $\mathbf{F}$, and $c \in \mathbf{F}(\sqrt{k})$, then $p(f(c)) = f(p(c))$.

(c) Conclude that if $c \in \mathbf{F}(\sqrt{k})$ is a root of p, then $f(c)$ is also a root of p.

(3) Show that $\sqrt{3} \notin \mathbf{Q}(\sqrt{2})$ and $\sqrt{2} \notin \mathbf{Q}(\sqrt{3})$.

(4) Suppose $p(x)$ is a polynomial with coefficients in $\mathbf{Q}$, and $p(\sqrt{2}+\sqrt{3}) = 0$. Use problems 2 and 3 above to show that $\sqrt{2} - \sqrt{3}$, $-\sqrt{2} + \sqrt{3}$, and $-\sqrt{2} - \sqrt{3}$ are also roots of p.

Deduce that the degree of p is at least 4.

(5) Find a polynomial $p(x)$ of degree 4, with coefficients in $\mathbf{Q}$, that has $\sqrt{2} + \sqrt{3}$ as a root. (Hint: by problem 4, you know all the roots of this polynomial.)