Math 124, problem set #3

due April 25, 2006

(1) Find the greatest common divisor (gcd) of the polynomials $f(x) = x^3 - 1$ and $g(x) = x^2 - 1$. Write the gcd as a linear combination of $f(x)$ and $g(x)$.

(2) Factor $x^6 - 1$ into irreducible polynomials in $\mathbf{Q}[x]$.

(3) Let $f(x) \in \mathbf{F}[x]$ be a polynomial of degree n. Show that there is an extension $\mathbf{E}$ of $\mathbf{F}$, with $[\mathbf{E} : \mathbf{F}] \leq n!$, such that $f(x)$ factors into linear factors in $\mathbf{E}[x]$.

(4) Suppose a, b are algebraic over a field $\mathbf{F}$.
   (a) Show that $[\mathbf{F}(a, b) : \mathbf{F}] \leq \deg_{\mathbf{F}}(a) \deg_{\mathbf{F}}(b)$.
   (b) Show that if $\deg_{\mathbf{F}}(a)$ is relatively prime to $\deg_{\mathbf{F}}(b)$, then
$$[\mathbf{F}(a, b) : \mathbf{F}] = \deg_{\mathbf{F}}(a) \deg_{\mathbf{F}}(b).$$
   (c) Show that if a and b are two different roots of $x^3 - 2$ then
$$[\mathbf{Q}(a, b) : \mathbf{Q}] \neq \deg_{\mathbf{Q}}(a) \deg_{\mathbf{Q}}(b).$$

(5) Suppose that $\mathbf{E}/\mathbf{F}$ is a field extension, and $[\mathbf{E} : \mathbf{F}] = n$. Prove that if $\alpha \in \mathbf{E}$, then $\mathbf{E} = \mathbf{F}(\alpha)$ if and only if $\deg_{\mathbf{F}}(\alpha) = n$.