Math 124, problem set #3
due April 25, 2006

(1) Find the greatest common divisor (gcd) of the polynomials \( f(x) = x^3 - 1 \) and \( g(x) = x^2 - 1 \). Write the gcd as a linear combination of \( f(x) \) and \( g(x) \).

(2) Factor \( x^6 - 1 \) into irreducible polynomials in \( \mathbb{Q}[x] \).

(3) Let \( f(x) \in \mathbb{F}[x] \) be a polynomial of degree \( n \). Show that there is an extension \( E \) of \( \mathbb{F} \), with \( [E : \mathbb{F}] \leq n! \), such that \( f(x) \) factors into linear factors in \( E[x] \).

(4) Suppose \( a, b \) are algebraic over a field \( \mathbb{F} \).
   (a) Show that \( [\mathbb{F}(a, b) : \mathbb{F}] \leq \deg_{\mathbb{F}}(a) \deg_{\mathbb{F}}(b) \).
   (b) Show that if \( \deg_{\mathbb{F}}(a) \) is relatively prime to \( \deg_{\mathbb{F}}(b) \), then
        \( [\mathbb{F}(a, b) : \mathbb{F}] = \deg_{\mathbb{F}}(a) \deg_{\mathbb{F}}(b) \).
   (c) Show that if \( a \) and \( b \) are two different roots of \( x^3 - 2 \) then
        \( [\mathbb{Q}(a, b) : \mathbb{Q}] \neq \deg_{\mathbb{Q}}(a) \deg_{\mathbb{Q}}(b) \).

(5) Suppose that \( E/\mathbb{F} \) is a field extension, and \( [E : \mathbb{F}] = n \). Prove that if \( \alpha \in E \), then \( E = \mathbb{F}(\alpha) \) if and only if \( \deg_{\mathbb{F}}(\alpha) = n \).