Math 124, problem set #4

due May 2, 2006

(1) (a) Find a number a so that $\mathbf{Q}(a) = \mathbf{Q}(\sqrt{2}, \sqrt{3})$. Justify your answer.
    (b) Show that $\mathbf{Q}(e^{2\pi i/35}) = \mathbf{Q}(e^{2\pi i/5}, e^{2\pi i/7})$. Give a complete and explicit justification.

(2) Let $\mathbf{F}$ be the smallest field with the property that $x^7 - 7$ factors into a product of linear factors in $\mathbf{F}[x]$. What is $[\mathbf{F} : \mathbf{Q}]$?

(3) Factor $x^{10} - 1$ into irreducible polynomials in $\mathbf{Q}[x]$. Explain why each factor is irreducible. What is $[\mathbf{Q}(e^{2\pi i/10}) : \mathbf{Q}]$?

(4) Suppose $\mathbf{F}$ is a field and p is a prime. Let $f(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$, and let $f(x) = \prod_{i=1}^{d} g_i(x)$ be the factorization of $f(x)$ into irreducible polynomials in $\mathbf{F}[x]$. Prove that the irreducible factors $g_i(x)$ all have the same degree. (Hint: if a_i is a root of g_i, show that $\mathbf{F}(a_i) = \mathbf{F}(a_j)$ for every i and j.)

(5) Suppose a is a root of the (irreducible) polynomial $x^4 + x^3 + x^2 + x + 1 \in \mathbf{Q}[x]$.
    (a) Show that $a + a^{-1}$ is a root of $x^2 + x - 1$.
    (b) Show that $[\mathbf{Q}(a + a^{-1}) : \mathbf{Q}] = 2$.
    (c) Show that $[\mathbf{Q}(a) : \mathbf{Q}(a + a^{-1})] = 2$.
    (d) Deduce that a is constructible.
    (e) Deduce that a regular pentagon is constructible