Math 124, problem set #7

due May 30, 2006

(1) Let $\mathbf{E} = \mathbf{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ and $\mathbf{F} = \mathbf{Q}(\sqrt{3})$.
   (a) Show that $\mathbf{E}/\mathbf{Q}$ is normal. (This should not be difficult.)
   (b) Describe $G(\mathbf{E}/\mathbf{Q})$ and $G(\mathbf{E}/\mathbf{F})$.
   (c) List all subgroups of $G(\mathbf{E}/\mathbf{F})$ and the corresponding fields $\mathbf{K}$, $\mathbf{F} \subset \mathbf{K} \subset \mathbf{E}$.

(2) Let $\mathbf{E}$ be the splitting field of the polynomial $f(x) = x^4 - 5 \in \mathbf{Q}[x]$.
   (a) Show that $G(\mathbf{E}/\mathbf{Q}) = D_8$, the dihedral group of order 8. (Recall that D_8 is the group generated by r and s, where $r^4 = 1$, $s^2 = 1$, and $rs = sr^{-1}$.)
   (b) List all intermediate fields $\mathbf{F}$, $\mathbf{Q} \subset \mathbf{F} \subset \mathbf{E}$. Which ones are normal extensions of $\mathbf{Q}$?

(3) Suppose $\mathbf{F}$ is a field, $a \in \mathbf{F}$, and a is not a cube in $\mathbf{F}$. Let $\omega = e^{2\pi i/3}$.
   (a) If $\omega \in \mathbf{F}$, show that $\mathbf{F}(\sqrt[3]{a})/\mathbf{F}$ is normal.
   (b) If $\omega \notin \mathbf{F}$, show that $\mathbf{F}(\sqrt[3]{a})/\mathbf{F}$ is *not* normal. (Hint: you will have to show that $\omega \notin \mathbf{F}(\sqrt[3]{a})$, this requires proof.)

(4) Show that if $\mathbf{E}/\mathbf{F}$ and $\mathbf{K}/\mathbf{F}$ are normal extensions, then $\mathbf{EK}/\mathbf{F}$ is a normal extension.

(5) Find a normal extension $\mathbf{E}/\mathbf{K}$ and a normal extension $\mathbf{K}/\mathbf{F}$ such that $\mathbf{E}/\mathbf{F}$ is *not* normal.