Math 124, problem set #8

due June 6, 2006

(1) Show that if G is a finite commutative group, then G is solvable.

(2) Show that D_{2n}, the dihedral group of order $2n$, is solvable.

(3) Show that if G is a solvable group, and H is a subgroup of G, then H is solvable.

(4) Suppose $\mathbf{F}$ is a field, $f(x) \in \mathbf{F}[x]$ is an irreducible polynomial, and r_1, r_2 are roots of f. Show that the fields $\mathbf{F}(r_1)$ and $\mathbf{F}(r_2)$ are isomorphic (i.e., there is a bijection $\phi : \mathbf{F}(r_1) \to \mathbf{F}(r_2)$ that preserves addition and multiplication).

(5) What is wrong with the following proof of the (false) statement that every normal extension $\mathbf{E}/\mathbf{F}$ is solvable?

"Proof". We proceed by induction on $[\mathbf{E} : \mathbf{F}]$. If $[\mathbf{E} : \mathbf{F}] = 1$, then $\mathbf{E} = \mathbf{F}$ and $\mathbf{E}/\mathbf{F}$ is solvable.

Suppose $[\mathbf{E} : \mathbf{F}] > 1$, and let p be a prime dividing $[\mathbf{E} : \mathbf{F}]$. Then $G(\mathbf{E}/\mathbf{F})$ has an element ϕ of order p. Let H be the subgroup of order p generated by ϕ, and let $\mathbf{K} = \mathbf{E}^H$. Then $\mathbf{E}/\mathbf{K}$ is normal of prime order p. Since $[\mathbf{K} : \mathbf{F}] < [\mathbf{E} : \mathbf{F}]$, by induction we know that $\mathbf{K}/\mathbf{F}$ is solvable. Therefore we have

$$\mathbf{F} = \mathbf{F}_0 \subset \mathbf{F}_1 \subset \cdots \subset \mathbf{F}_N = \mathbf{K}$$

with each $\mathbf{F}_{i+1}/\mathbf{F}_i$ normal of prime degree. Now the tower

$$\mathbf{F} = \mathbf{F}_0 \subset \mathbf{F}_1 \subset \cdots \subset \mathbf{F}_N = \mathbf{K} \subset \mathbf{F}_{N+1} = \mathbf{E}$$

shows that $\mathbf{E}/\mathbf{F}$ is solvable. $\square$