Pre-Putnam Exam Solutions

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1. Find all polynomials $p(x)$ with real coefficients satisfying the differential equation

$$7\frac{d}{dx}[xp(x)] = 3p(x) + 4p(x+1), \qquad -\infty < x < \infty.$$

Solution:

Suppose we have a solution of degree n, so that $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. By looking at the coefficient of x^n, we have $7(n+1)a_n = 3a_n + 4a_n$, which implies $n = 0$ (in assuming that our polynomial has degree n, we have assumed $a_n \neq 0$). Then, we see that for any constant $c, p(x) = c$ satisfies our differential equation.

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2. Show that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$$

for all positive integers n.

Solution:

We proceed by induction on n. Notice $1 < 2\sqrt{1}$. Define

$$f(n) = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}},$$

and assume as our induction hypothesis that $f(k) < 2\sqrt{k}$. Consider

$$(*) \qquad f(k+1) = f(k) + \frac{1}{\sqrt{k+1}} < 2\sqrt{k} + \frac{1}{\sqrt{k+1}}.$$

Since we would like an expression on the right involving $2\sqrt{k+1}$, it is natural to consider

$$2(\sqrt{k+1} - \sqrt{k}) = 2\frac{(\sqrt{k+1} - \sqrt{k})(\sqrt{k+1} + \sqrt{k})}{\sqrt{k+1} + \sqrt{k}} = 2\frac{1}{\sqrt{k+1} + \sqrt{k}} > \frac{1}{\sqrt{k+1}}.$$

Using this on the right-hand side of (*), we have $f(k+1) < 2\sqrt{k+1}$.

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3. Show that

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq 3$$

for all positive real numbers x, y, and z.

Solution:

Using the Arithmetic Mean - Geometric Mean inequality, we have

$$\frac{\frac{x}{y} + \frac{y}{z} + \frac{z}{x}}{3} \geq \sqrt[3]{\frac{x}{y}\frac{y}{z}\frac{z}{x}} = 1,$$

and our result follows.

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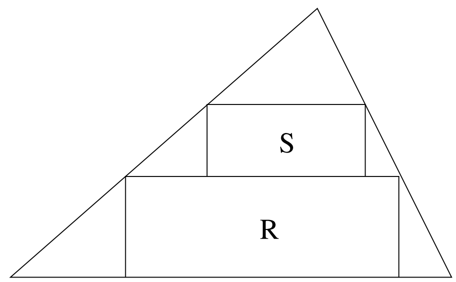
4. Let T be an acute triangle. Inscribe a pair of rectangles R and S in T as shown in the figure below. Let $A(X)$ denote the area of any polygon X. Find the maximum value of $\dfrac{A(R)+A(S)}{A(T)}$, where T ranges over all acute triangles, and R and S range over all inscribed rectangles.



Solution:

Let b_R, h_R, b_S, h_S, b_T, and h_T be the lengths of the bases and heights of R, S, and T, respectively. Observing the similarity of three triangles, we have

$$\frac{b_T}{h_T} = \frac{b_R}{h_T - h_R} = \frac{b_S}{h_T - h_R - h_S}.$$

Thus

$$A(S) = b_S h_S = \frac{b_T}{h_T}(h_T - h_R - h_S)h_S,$$

and for any fixed R and T, this area is maximized when $h_S = \frac{1}{2}(h_T - h_R)$ (see this using properties of parabolas or the first derivative test). Also,

$$A(R) = b_R h_R = \frac{b_T}{h_T}(h_T - h_R)h_R.$$

Thus, for any fixed R and T, if we choose $A(S)$ to be maximal,

$$\frac{A(R)+A(S)}{A(T)} = \frac{\frac{b_T}{h_T}(h_T - h_R)h_R + \frac{b_T}{h_T}(\frac{h_T - h_R}{2})^2}{\frac{1}{2}b_T h_T}.$$

For any fixed T, this quantity is maximized when $h_R = \frac{1}{3}h_T$ (again using properties of parabolas or the first derivative test). Substituting this in for h_R, we see that the maximum value of $\dfrac{A(R)+A(S)}{A(T)}$ is $\frac{2}{3}$, independent of T.

Is there a better solution?

5. Let $a_1, a_2, \ldots, a_{100}$ be integers. Show that there exist i, j, k, and l with $i \neq j$ and $i \neq l$ such that $a_i - a_j + a_k - a_l$ is a multiple of 2004.

Solution:

Consider the multiset $S = \{a_n + a_m | 1 \leq n < m \leq 100\}$. This multiset has $\binom{100}{2} = 4950 > 2004$ elements, and so by the pigeonhole principle, two of these elements must be congruent modulo 2004. Let these two elements be $a_c + a_d$ and $a_j + a_l$. Since $\{c, d\} \neq \{j, l\}$, one element of $\{c, d\}$ is $\notin \{j, l\}$; set i to be that element of $\{c, d\}$, and k to be the other. Then $a_i - a_j + a_k - a_l$ is a multiple of 2004, with $i \neq j$ and $i \neq l$.

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6. Find all real valued functions $F(x)$ defined for all real $x \neq 0, 1$ satisfying the functional equation

$$F(x) + F\left(\frac{x-1}{x}\right) = 1 + x.$$

Solution:

Notice that for any $x \neq 0, 1$,

$$\begin{aligned} F(x) + F\left(\frac{x-1}{x}\right) &= 1 + x, \\ F\left(\frac{x-1}{x}\right) + F\left(\frac{1}{1-x}\right) &= 1 + \frac{x-1}{x}, \quad \text{and} \\ F\left(\frac{1}{1-x}\right) + F(x) &= 1 + \frac{1}{1-x}. \end{aligned}$$

Thus

$$\begin{aligned} 2F(x) &= F(x) + F\left(\frac{x-1}{x}\right) - F\left(\frac{x-1}{x}\right) - F\left(\frac{1}{1-x}\right) + F\left(\frac{1}{1-x}\right) + F(x) \\ &= 1 + x - \left(1 + \frac{x-1}{x}\right) + 1 + \frac{1}{1-x} \\ &= 1 + x + \frac{1-x}{x} + \frac{1}{1-x}, \end{aligned}$$

and

$$F(x) = \frac{1}{2}\left(1 + x + \frac{1-x}{x} + \frac{1}{1-x}\right).$$
