Pre-Putnam Exam

This exam was designed to be taken in 3 hours without notes, books, calculators, collaboration, or interruption. Good luck.

1. Find all polynomials $p(x)$ with real coefficients satisfying the differential equation

$$7\frac{d}{dx}[xp(x)] = 3p(x) + 4p(x+1), \qquad -\infty < x < \infty.$$

2. Show that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$$

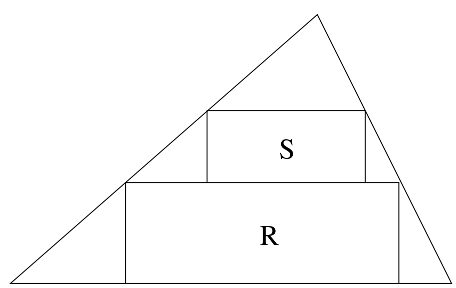
for all positive integers n.

3. Show that

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq 3$$

for all positive real numbers x, y, and z.

4. Let T be an acute triangle. Inscribe a pair of rectangles R and S in T as shown in the figure below. Let $A(X)$ denote the area of any polygon X. Find the maximum value of $\dfrac{A(R)+A(S)}{A(T)}$, where T ranges over all acute triangles, and R and S range over all inscribed rectangles.



5. Let $a_1, a_2, \ldots, a_{100}$ be integers. Show that there exist i, j, k, and l with $i \neq j$ and $i \neq l$ such that $a_i - a_j + a_k - a_l$ is a multiple of 2004.

6. Find all real valued functions $F(x)$ defined for all real $x \neq 0, 1$ satisfying the functional equation

$$F(x) + F\left(\frac{x-1}{x}\right) = 1 + x.$$