Math 232
Problem set #2

(1) Let $K = \mathbb{Q}(\sqrt[3]{5})$. You may assume that $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{5}]$. For every rational prime $p$, show how $p\mathcal{O}_K$ factors into prime ideals in $\mathcal{O}_K$, and for each prime ideal factor $\mathfrak{p}$ of $p\mathcal{O}_K$ find $|\mathcal{O}_K/\mathfrak{p}|$.

(2) Suppose $K \subset L \subset F$ are number fields, and $F/K$ is Galois. Fix a prime $\mathfrak{p}$ of $K$, a prime $\mathfrak{p}_L$ of $L$ above $\mathfrak{p}$, and a prime $\mathfrak{p}_F$ of $F$ above $\mathfrak{p}_L$. Write $G_{\mathfrak{p}_F}(F/K)$ for the decomposition group of $\mathfrak{p}_F$ in $\text{Gal}(F/K)$, $I_{\mathfrak{p}_F}(F/K)$ for the inertia group of $\mathfrak{p}_F$ in $\text{Gal}(F/K)$, etc.

(a) Show that $G_{\mathfrak{p}_L}(L/K) = G_{\mathfrak{p}_F}(F/K) \cap \text{Gal}(F/L)$ and $I_{\mathfrak{p}_L}(L/K) = I_{\mathfrak{p}_F}(F/K) \cap \text{Gal}(F/L)$.

(b) If $L/K$ is Galois, show that $G_{\mathfrak{p}_L}(L/K)$ (resp. $I_{\mathfrak{p}_L}(L/K)$) is the image of $G_{\mathfrak{p}_F}(F/K)$ (resp. $I_{\mathfrak{p}_F}(F/K)$) in $\text{Gal}(L/K)$.

(3) Let $K = \mathbb{Q}(\sqrt{-5})$. Let $\mathfrak{p}$ be the ideal generated by 2 and $1 + \sqrt{-5}$.

(a) Show that $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_K$.

(b) Give generators for the fractional ideal $\mathfrak{p}^{-1}$.

(4) Let $L = \mathbb{Q}(\sqrt{-3}, \sqrt[3]{5})$, the splitting field of $x^3 - 5$. If $p$ is a rational prime, describe as best you can the splitting of $p$ in $L$ (for example, the number of primes of $L$ above $p$, and the decomposition groups and inertia groups of those primes). You might find it helpful to use problems (1) and (2), and problem (6) from problem set #1.