(1) Suppose $K \subset F \subset L$ are number fields, $p_K$ is a prime of $K$, $p_F$ is a prime of $F$ above $p_K$, and $p_L$ is a prime of $L$ above $p_F$.
(a) Show that 
$$e(p_L/p_K) = e(p_L/p_F)e(p_F/p_K), \quad f(p_L/p_K) = f(p_L/p_F)f(p_F/p_K).$$
(b) If $a$ is a fractional ideal of $L$, show that $N_{L/K}a = N_{F/K}(N_{L/F}(a))$.

In the following problems, $K$ is a number field, $p$ is a rational prime, $p$ is a prime of $K$ above $p$, and $|\cdot|_p$, $||\cdot||_p$ are the absolute values on $K$ extending $|\cdot|_p$, normalized so that

\[*\] $$|p|_p = 1/p, \quad ||x||_p = \#(O_K/p)^{-\text{ord}_p(x)}.$$

(2) For which real number $c$ is $||x||_p = |x|^c_p$?

(3) Suppose $K$ is a number field, and $|\cdot|$ is an absolute value whose restriction to $\mathbb{Q}$ is $|\cdot|_p$.
(a) Show that $|x| \leq 1$ for every $x \in O_K$.
(b) Show that there is a prime $p$ of $K$ above $p$ such that $|\cdot| = |\cdot|_p$.

(4) Show that if $x \in \mathbb{Q}$, then $\prod_{p \leq \infty} |x|_p = 1$.

(5) Suppose $K$ is a number field, Galois over $\mathbb{Q}$. Show using \[*\] that if $x \in K$ and $p$ is a rational prime, then 
$$\prod_{p | p} ||x||_p = |N_{K/\mathbb{Q}}x|_p$$

(6) Use the proof of Hensel’s Lemma to find a rational number $r$ such that 
$$|r^2 - 2|_7 < .001$$

(7) Suppose $m$ is a positive integer. Prove that $\mathbb{Q}_p$ contains a primitive $m$-th root of unity if and only if $m \equiv 1 \pmod{p}$. 