Math 194, problem set #3

For discussion Tuesday, October 21, 2008

(1) For which integers n is $(n^3 - 3n^2 + 4)/(2n - 1)$ an integer?
(Andreescu & Gelca)

(2) Is it possible to place 1995 different positive integers around a circle so that for any two adjacent numbers, the ratio of the larger to the smaller is a prime?
(Moscow Mathematical Olympiad)

(3) Let p be a prime number. Prove that there are infinitely many multiples of p whose last 10 digits are all distinct. (International Mathematical Olympiad)

(4) If the last 4 digits of a perfect square are equal, prove that they are all zero.
(Andreescu & Gelca)

(5) Prove that the sequence $2^n - 3$, $n \geq 2$, contains an infinite subsequence whose terms are pairwise relatively prime. (Andreescu & Gelca)

(6) If n, a, b are positive integers, show that $\gcd(n^a - 1, n^b - 1) = n^{\gcd(a,b)} - 1$.

(7) We say that a lattice point $(x, y) \in \mathbf{Z}^2$ is *visible from the origin* if x and y are relatively prime. Prove that for every positive integer n there is a lattice point (a, b) whose distance from every visible point is greater than n.
(American Mathematical Monthly 1977)

(8) Prove that there is no integer that is doubled when the first (leftmost) digit is transferred to the end. (USSR Olympiad)

(9) Fix an integer $b \geq 3$. Let $f(1) = 1$, and for each $n \geq 2$, define $f(n) = nf(d)$, where d is the number of base-b digits of n. Show that the sum

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

diverges. (part of A-6, Putnam 2002)

(10) If n is a positve integer, prove that $n!$ is not divisible by 2^n.
(Mathematics Competition, Soviet Union 1971)