1. Suppose \( A \) and \( B \) are two \( n \times n \) matrices that commute (i.e., \( AB = BA \)), and for some positive integers \( r, s \) we have \( A^r = I \) (the \( n \times n \) identity matrix) and \( B^s = 0 \) (the \( n \times n \) zero matrix). Prove that \( A + B \) is invertible, and find its inverse. 

(Andreescu & Gelca)

2. Suppose \( a, b \geq 2 \) are relatively prime integers. For every \( n \geq 0 \) show that \( a^{2n} + b^{2n} \) is not divisible by \( a + b \). 

(Gelca)

3. Find all polynomials \( P(x) \) with real coefficients such that 
\[
(x + 1)P(x) = (x - 2)P(x + 1)
\]

4. Let \( P(x) \) be a polynomial of odd degree with real coefficients. Show that the equation \( P(P(x)) = 0 \) has at least as many real roots as the equation \( P(x) = 0 \), (counted without multiplicities). 

(Russian Mathematical Olympiad, 2002)

5. Given an integer \( n \geq 1 \), find all polynomials \( P(x) \) with real coefficients such that \( P(x)^n = P(x^n) \) for all \( x \).

6. Suppose you have a calculator, but the multiplication and division buttons are broken. You can add, subtract, and take the inverse of a number, but you can’t multiply or divide. Show how to find the product of (any) two numbers, using at most 20 operations. 

(Quantum)

7. Let \( S \) be the smallest set of rational functions (i.e., ratios of polynomials) in the variables \( x \) and \( y \) with real coefficients, containing \( f(x, y) = x \) and \( g(x, y) = y \) and closed under addition, subtraction, and taking reciprocals. Show that \( S \) does not contain the constant function \( h(x, y) = 1 \). 

(American Mathematical Monthly, 1987)

8. Suppose \( f(x) \) is a polynomial with integer coefficients, and for some integer \( k \) there are \( k \) consecutive integers \( n, n + 1, \ldots, n + k - 1 \) such that none of the values \( f(n), f(n + 1), \ldots, f(n + k - 1) \) are divisible by \( k \). Prove that \( f(x) \) has no integer roots. 

(Putnam, 1940)

9. Suppose that \( a \) and \( b \) are different roots of \( x^3 + x - 1 \). Prove that \( ab \) is a root of \( x^3 - x^2 - 1 \).

10. Show that there are infinitely many positive integers \( a \) such that \( n^4 + a \) is not prime for any natural number \( n \).