Math 194
For discussion Tuesday, Nov. 18, 2008

You are *strongly urged* to write up and hand in a careful and complete solution to (at least) one of these problems.

1. Evaluate
$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{k^2 + n^2}}.$$

(Larson 6.8.5)

2. Let $f(x) = \sum_{k=1}^{n} a_k \sin(kx)$ with $a_i \in \mathbf{R}$, $n \geq 1$. Prove that if $f(x) \leq |\sin(x)|$ for every x, then
$$\left| \sum_{k=1}^{n} k a_k \right| \leq 1.$$

(Putnam 1967)

3. Let $f(x)$ be a continuous function on $[0, 1]$ such that $f(0) = f(1) = 0$ and $2f(x) + f(y) = 3f(\frac{2x+y}{3})$ for all $x, y \in [0, 1]$. Prove that $f(x) = 0$ for all $x \in [0, 1]$. (Vietnamese Mathematical Olympiad, 1999)

4. Suppose $f : \mathbf{R} \to \mathbf{R}$ is a continuous function such that $|f(x) - f(y)| \geq |x - y|$ for every $x, y \in \mathbf{R}$. Show that the range of f is $\mathbf{R}$, i.e., for every $c \in \mathbf{R}$ there is an x such that $f(x) = c$.

(De Souza & Silva, *Berkeley Problems in Mathematics*)

5. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a continuous function, and define
$$g(x) = f(x) \int_0^x f(t)dt.$$

Prove that if g is a nonincreasing function, then $f(x) = 0$ for every x.

(Romanian Olympiad 1978)

6. Let $f : [0, 1] \to \mathbf{R}$ be a function with a continuous derivative, such that $f(0) = 0$ and $0 < f'(x) \leq 1$ for every x. Show that
$$\left(\int_0^1 f(x)dx \right)^2 \geq \int_0^1 (f(x))^3 dx.$$

(Putnam 1973)

7. Suppose f and g are n-times continuously differentiable functions in a neighborhood of a point a, such that $f(a) = g(a)$, $f'(a) = g'(a)$, ..., $f^{(n-1)}(a) = g^{(n-1)}(a)$, and $f^{(n)}(a) \neq g^{(n)}(a)$. Evaluate

$$\lim_{x \to a} \frac{e^{f(x)} - e^{g(x)}}{f(x) - g(x)}.$$

(N. Georgescu-Roegen)

8. Let n be a positive integer, and $f : [a, b] \to \mathbf{R}$ a continuous function, n-times differentiable on (a, b). Prove that if the graph of f has $n + 1$ collinear points, then there is a point $c \in (a, b)$ such that $f^{(n)}(c) = 0$.

(G. Sireţchi, *Mathematics Gazette, Bucharest*)

9. Suppose $x_1, x_2, \ldots, x_n \in \mathbf{R}$. Find the real number(s) a that minimize the expression

$$|a - x_1| + |a - x_2| + \cdots + |a - x_n|.$$

(Andreescu & Gelca)

10. Prove that for every natural number $n \geq 2$ and every $x \in [-1, 1]$,

$$(1 + x)^n + (1 - x)^n \leq 2^n.$$

(*Exercises and Problems in Algebra*, Bucharest 1983)