Math 194, problem set #7
For discussion Tuesday November 25

You are *strongly urged* to write up and hand in a careful and complete solution to (at least) one of these problems.

1. Prove for \( n \geq 1 \) that
\[
\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}. \tag{Engel}
\]

2. Let \( a_1/b_1, a_2/b_2, \ldots, a_n/b_n \) be \( n \) fractions with \( b_i > 0 \) for \( i = 1, 2, \ldots, n \). Show that the fraction
\[
\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n}
\]
is a number between the largest and smallest of these fractions. \tag{Larson 7.1.10}

3. Prove that \( \sqrt{n} < 1 + \sqrt{2/n} \) if \( n \) is a positive integer. \tag{Larson 7.1.15}

4. Prove that for every integer \( n \geq 2 \),
\[
\prod_{k=1}^{n} \left( \frac{n}{k} \right) \leq \left( \frac{2^n - 2}{n - 1} \right)^{n-1}. \tag{Storey}
\]

5. Prove that if \( a_1, \ldots, a_n \) are real numbers and \( a_1 + \cdots + a_n = 1 \), then
\[
a_1^2 + \cdots + a_n^2 \geq 1/n.
\]

6. Prove that the sequence \( \{a_n\} \) defined by
\[
a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log(n)
\]
converges.
7. Show that for all $x$,
\[ 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2n}}{(2n)!} > 0. \]

8. Given a point $(a, b)$ with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at $(a, b)$, one on the $x$-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

(Putnam, 1998)

9. Prove that for every positive $n$, $n! > \left(\frac{n}{e}\right)^n$.

(Larson 7.1.12)

10. Prove that
\[ \left(\frac{a + 1}{b + 1}\right)^{b+1} > \left(\frac{a}{b}\right)^b \]
    for every $a, b > 0$, $a \neq b$.

(Larson 7.4.17)

11. Find all positive integers $n, k_1, k_2, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and
\[ \frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1. \]

(Putnam 2005)