Math 232

Problem set #3, due Monday February 11

In problems (1)–(3), suppose $p$ is a prime number.

(1) List all subgroups of $\mathbb{Q}_p^\times$ of index 2.

(2) List all quadratic extensions of $\mathbb{Q}_p$.

(3) Suppose $n \in \mathbb{Z}^+$, and $\ell$ is a rational prime (possibly equal to $p$). If $x \in \mathbb{Q}_\ell^\times$, compute the Artin symbol

$$[x, \mathbb{Q}(\mu_{p^n})/\mathbb{Q}] \in \text{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q}) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times$$

where we view $x \in \mathbb{J}_\mathbb{Q}$ via the embedding $\mathbb{Q}_\ell^\times \subset \mathbb{J}_\mathbb{Q}$.

(4) Suppose $a$ is an ideal of the number field $K$, and let $K(a)$ denote the ray class field of $K$ modulo $a$. Describe the Galois group $\text{Gal}(K(a)/K)$ (relate it to $\mathcal{O}_K/a$.) What is the degree $[K(a) : K]$ when $K$ is an imaginary quadratic field?

(5) (a) Prove that $\mathbb{Q}(\sqrt{5}, \sqrt{13})$ is an everywhere-unramified quadratic extension of $\mathbb{Q}(\sqrt{65})$.

(b) The ideal class group of $\mathbb{Q}(\sqrt{65})$ is 2 (you may assume this). Deduce that $\mathbb{Q}(\sqrt{5}, \sqrt{13})$ is the Hilbert class field of $\mathbb{Q}(\sqrt{65})$.

(c) Show that if $m$ is a squarefree product of $d$ odd primes, then the ideal class group of $\mathbb{Q}(\sqrt{d})$ has order at least $2^{d-1}$. (Hint: find an unramified abelian extension of degree $2^{d-1}$.)