Math 232B
Problem set #5, due March 3, 2008

(1) Suppose \( K \) is a local field, \( \mathcal{O} \) is its ring of integers, \( \mathfrak{p} \) is the maximal ideal of \( \mathcal{O} \), \( k = \mathcal{O}/\mathfrak{p} \) is the residue field, \( p \) is the characteristic of \( k \), and \( F \) is a formal group over \( \mathcal{O} \). For every \( n \geq 1 \), let \( F(p^n) \) denote the set \( \mathfrak{p}^n \) with the group structure defined by \( F \).

(a) Prove that \( F(p^n)/F(p^{n+1}) \cong k \).

(b) Prove that \( F(\mathfrak{p}) \) contains no prime-to-\( p \) torsion.

(c) Prove that for \( n \) sufficiently large, \( F(p^n) \) is isomorphic to \( \mathbb{Z}_p^d \) for some \( d \) (hint: use the isomorphism from \( F \) to the additive formal group). How large does \( n \) have to be? What is \( d \)?

(2) Suppose \( k \) is a field of characteristic \( p \) and \( F \) is a formal group over \( k \). Let \([p](X) \in k[[X]]\) be the endomorphism of \( F \) “multiplication by \( p \)” (i.e., \([p](X) = F(F(F(\ldots F(X,X),X,\ldots),X)\) with \( p \) \( X \)'s).

(a) Show that there is an integer \( h \), \( 1 \leq h \leq \infty \) such that \([p](X) \equiv uXp^h \pmod{\deg p^h + 1}\) for some \( u \in k^\times \). The integer \( h \) is called the height of \( F \).

(b) Exhibit formal groups of height 1 and \( \infty \).

(3) Suppose \( K \) is a local field, \( \pi \) is a uniformizing parameter of \( K \), \( k \) is the residue field of \( K \), and \( f \in \mathcal{F}_\pi \). Let \( \tilde{F}_f \) denote the formal group over \( k \) that is the reduction of \( F_f \). What is the height of \( \tilde{F}_f \)?