# Math 232C

Problem set #1, due April 9, 2008

If you are stuck on any of the first three problems after trying them for a while, ask me for a hint.

(1) Prove that if $G$ is a finite group and $A$ is a $G$-module, then there is a $G$-module isomorphism
$$\mathrm{Hom}(\mathbf{Z}[G], A) \cong \mathbf{Z}[G] \otimes A.$$
Here $\gamma \in G$ acts on $\mathrm{Hom}(\mathbf{Z}[G], A)$ by $f^\gamma(x) = \gamma f(\gamma^{-1}x)$, and on $\mathbf{Z}[G] \otimes A$ by $\gamma(x \otimes a) = (\gamma x) \otimes (\gamma a)$.

(2) Suppose $G$ is a group and $A$ is a $G$-module, and let $A_0$ be the $G$-module whose underlying abelian group is that of $A$, but with trivial $G$-action. Show that there is a $G$-module isomorphism
$$\mathrm{Hom}(\mathbf{Z}[G], A) \cong \mathrm{Hom}(\mathbf{Z}[G], A_0).$$

(3) If $G$ is a group and $A$ is a $G$-module, show that there is an injective $G$-module homomorphism
$$A \hookrightarrow \mathrm{Hom}(\mathbf{Z}[G], A).$$

(4) Suppose
$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$
is an exact sequence of abelian groups, and $D$ is an abelian group. Show that the induced sequence
$$0 \longrightarrow \mathrm{Hom}(C, D) \longrightarrow \mathrm{Hom}(B, D) \longrightarrow \mathrm{Hom}(A, D)$$
is exact.

(5) Suppose $G$ is a finite cyclic group, with generator $\gamma$. Show that the complex
$$\cdots \xrightarrow{d_{i+1}} \mathbf{Z}[G] \xrightarrow{d_i} \mathbf{Z}[G] \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_2} \mathbf{Z}[G] \xrightarrow{d_1} \mathbf{Z}[G] \xrightarrow{d_0} \mathbf{Z} \to 0$$
is exact, where $d_i$ is multiplication by $\gamma - 1$ if $i$ is odd, multiplication by $\sum_{g \in G} g$ if $i > 0$ is even, and $d_0$ is the map that sends every $g \in G$ to 1.