Math 232C

Problem set #2, due April 16, 2008

(1) Suppose K is a field of characteristic p, K^{sep} is its separable closure, and $G = \text{Gal}(K^{\text{sep}}/K)$.

  (a) Show that the map $\wp : K^{\text{sep}} \to K^{\text{sep}}$ defined by $\wp(x) = x^p - x$ is a surjective homomorphism of G-modules, with kernel equal to $\mathbf{F}_p$.

  (b) Using (a), show that $H^1(G, \mathbf{Z}/p\mathbf{Z}) \cong K/\wp(K)$.

  (c) Using (a), show that there is a bijection between the set of abelian extensions of K with Galois group killed by p, and the set of subgroups of $K/\wp(K)$.

(2) Suppose F/K is a finite Galois extension of fields. Let $G_K = \text{Gal}(K^{\text{sep}}/K)$, and $G_F = \text{Gal}(K^{\text{sep}}/F) \subset G_K$. If A is a topological G_K-module, use the inflation-restriction exact sequence for finite groups to show that

$$0 \longrightarrow H^1(\text{Gal}(F/K), A^{G_F}) \xrightarrow{\text{Inf}} H^1(G_K, A) \xrightarrow{\text{Res}} H^1(G_F, A)$$

is exact.

(3) Recall that a G-module is co-induced if it is of the form $\text{Hom}(\mathbf{Z}[G], X)$ for some X. Show that if A is a co-induced G-module, and H is a subgroup of finite index in G, then A is a co-induced H-module. (Hint: show that there is an isomorphism of H-modules $\text{Hom}(\mathbf{Z}[G], X) \cong \text{Hom}(\mathbf{Z}[H], \mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} X)$).

(4) Suppose H is a subgroup of finite index in G, and A is a G-module.

  (a) Show that the map

$$a \mapsto \sum_{g \in G/H} ga$$

  is a well-defined homomorphism $\text{Cor} : H^0(H, A) \to H^0(G, A)$ ("Cor" for "corestriction").

  (b) Using dimension-shifting (recall that every G-module A can be embedded in a co-induced G-module) show that the map Cor of (a) induces a map $\text{Cor} : H^i(H, A) \to H^i(G, A)$ for every $i \geq 0$.