Math 232C
Problem set #2, due April 16, 2008

(1) Suppose $K$ is a field of characteristic $p$, $K^{\text{sep}}$ is its separable closure, and $G = \text{Gal}(K^{\text{sep}}/K)$.

(a) Show that the map $\varphi : K^{\text{sep}} \to K^{\text{sep}}$ defined by $\varphi(x) = x^p - x$ is a surjective homomorphism of $G$-modules, with kernel equal to $F_p$.

(b) Using (a), show that $H^1(G, \mathbb{Z}/p\mathbb{Z}) \cong K/\varphi(K)$.

(c) Using (a), show that there is a bijection between the set of abelian extensions of $K$ with Galois group killed by $p$, and the set of subgroups of $K/\varphi(K)$.

(2) Suppose $F/K$ is a finite Galois extension of fields. Let $G_K = \text{Gal}(K^{\text{sep}}/K)$, and $G_F = \text{Gal}(K^{\text{sep}}/F) \subset G_K$. If $A$ is a topological $G_K$-module, use the inflation-restriction exact sequence for finite groups to show that

$$0 \longrightarrow H^1(\text{Gal}(F/K), A^{G_F}) \xrightarrow{\text{Inf}} H^1(G_K, A) \xrightarrow{\text{Res}} H^1(G_F, A)$$

is exact.

(3) Recall that a $G$-module is co-induced if it is of the form $\text{Hom}(\mathbb{Z}[G], X)$ for some $X$. Show that if $A$ is a co-induced $G$-module, and $H$ is a subgroup of finite index in $G$, then $A$ is a co-induced $H$-module. (Hint: show that there is an isomorphism of $H$-modules $\text{Hom}(\mathbb{Z}[G], X) \cong \text{Hom}(\mathbb{Z}[H], \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} X)$).

(4) Suppose $H$ is a subgroup of finite index in $G$, and $A$ is a $G$-module.

(a) Show that the map

$$a \mapsto \sum_{g \in G/H} ga$$

is a well-defined homomorphism $\text{Cor} : H^0(H, A) \to H^0(G, A)$ (“Cor” for “corestriction”).

(b) Using dimension-shifting (recall that every $G$-module $A$ can be embedded in a co-induced $G$-module) show that the map $\text{Cor}$ of (a) induces a map $\text{Cor} : H^i(H, A) \to H^i(G, A)$ for every $i \geq 0$. 