Math 232C

Problem set #4, due April 30, 2008

(1) Prove that if G is a profinite group, and A is a discrete G-module (with continuous G-action), then $H^i(G, A)$ is a torsion group for every $i \geq 1$.

(2) Let $G = \mathbf{Z}_p$, and let A be a torsion G-module. We know that G has cohomological dimension 1, so in particular $H^3(G, A) = 0$. But it can happen that $H^1(G, A) \neq 0$. So what's wrong with the following argument?

Let $G = \mathbf{Z}_p$, and let A be a torsion G-module. If H is an open subgroup of G, then G/H is cyclic, so $H^3(G/H, A^H) \cong H^1(G/H, A^H)$. Therefore

$$H^3(G, A) = \varinjlim H^3(G/H, A^H) \cong \varinjlim H^1(G/H, A^H) = H^1(G, A)$$

(direct limits over open subgroups H of G).

(3)
- (a) Suppose that G is a finite group and A is a G-module. Suppose further that there is a γ in the center of G such that $\gamma - 1 : A \to A$ is bijective. Prove that $H^1(G, A) = 0$. (Hint: if c is a 1-cocycle, use the fact that $c(\gamma g) = c(g\gamma)$ for every $g \in G$.)
- (b) Use (a) to show that $H^1(\mathrm{GL}_n(\mathbf{F}_p), \mathbf{F}_p^n) = 0$, where $\mathrm{GL}_n(\mathbf{F}_p)$ acts on $\mathbf{F}_p^n$ in the natural way.