Math 232C

Problem set #5, due May 9, 2008

(1) (Shapiro's Lemma). Suppose G is a finite group, H is a subgroup of G, and A is an H-module. Define

$$B := \mathrm{Hom}_{\mathbf{Z}[H]}(\mathbf{Z}[G], A),$$

which we view as a G-module by $\phi^g(\alpha) = \phi(\alpha g)$. Prove that $H^i(G, B) = H^i(H, A)$ for every $i > 0$.

(Hint: Suppose $\cdots \to P_{i_1} \to P_i \to \cdots \to P_0 \to 0$ is a resolution with free $\mathbf{Z}[G]$-modules P_i. Then the P_i are also free $\mathbf{Z}[H]$-modules, so the P_i can be used to compute both $H^i(G, B)$ and $H^i(H, A)$. Now show that there are natural isomorphisms $\mathrm{Hom}_{\mathbf{Z}[G]}(P_i, B) \cong \mathrm{Hom}_{\mathbf{Z}[H]}(P_i, A)$.)

(2) Formulate and prove a version of (1) when G is a profinite group.

(3) Suppose F/K is a finite Galois extension of fields, $G = \mathrm{Gal}(L/K)$, and v is a place of K. Fix a place w of F above K, and let $G_w \subset G$ be the decomposition group of w. Use (1) to prove that

$$H^i(G, \prod_{u|v} F_u^\times) = H^i(G_w, F_w^\times)$$

for every $i \geq 0$. Similarly, if v is nonarchimedean, show that

$$H^i(G, \prod_{u|v} \mathcal{O}_{F_u}^\times) = H^i(G_w, \mathcal{O}_{F_w}^\times).$$