Math 194, problem set #3
For discussion Tuesday, October 19, 2010

(1) For which integers $n$ is $(n^3 - 3n^2 + 4)/(2n - 1)$ an integer?  
(Andreescu & Gelca)

(2) Is it possible to place 1995 different positive integers around a circle so that for any two adjacent numbers, the ratio of the larger to the smaller is a prime?  
(Moscow Mathematical Olympiad)

(3) Let $p$ be a prime number. Prove that there are infinitely many multiples of $p$ whose last 10 digits are all distinct.  
(International Mathematical Olympiad)

(4) If the last 4 digits of a perfect square are equal, prove that they are all zero.  
(Andreescu & Gelca)

(5) Prove that the sequence $2^n - 3, n \geq 2$, contains an infinite subsequence whose terms are pairwise relatively prime.  
(Andreescu & Gelca)

(6) If $n, a, b$ are positive integers, show that $\gcd(n^a - 1, n^b - 1) = n^{\gcd(a,b)} - 1$.

(7) We say that a lattice point $(x, y) \in \mathbb{Z}^2$ is visible from the origin if $x$ and $y$ are relatively prime. Prove that for every positive integer $n$ there is a lattice point $(a, b)$ whose distance from every visible point is greater than $n$.  
(American Mathematical Monthly 1977)

(8) Prove that there is no integer that is doubled when the first (leftmost) digit is transferred to the end.  
(USSR Olympiad)

(9) Fix an integer $b \geq 3$. Let $f(1) = 1$, and for each $n \geq 2$, define $f(n) = nf(d)$, where $d$ is the number of base-$b$ digits of $n$. Show that the sum

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

diverges.  
(part of A-6, Putnam 2002)

(10) If $n$ is a positive integer, prove that $n!$ is not divisible by $2^n$.  
(Mathematics Competition, Soviet Union 1971)