Math 194
For discussion Tuesday, Nov. 20, 2012

You are strongly urged to write up and hand in a careful and complete solution to (at least) one of these problems.

1. Evaluate
\[ \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{\sqrt{k^2 + n^2}}. \]

(Larson 6.8.5)

2. Let \( f(x) = \sum_{k=1}^{n} a_k \sin(kx) \) with \( a_i \in \mathbb{R}, \ n \geq 1. \) Prove that if \( f(x) \leq |\sin(x)| \) for every \( x, \) then
\[ \left| \sum_{k=1}^{n} ka_k \right| \leq 1. \]

(Putnam 1967)

3. Let \( f(x) \) be a continuous function on \([0, 1]\) such that \( f(0) = f(1) = 0 \) and \( 2f(x) + f(y) = 3f\left(\frac{2x+y}{3}\right) \) for all \( x, y \in [0, 1]. \) Prove that \( f(x) = 0 \) for all \( x \in [0, 1]. \)

(Vietnamese Mathematical Olympiad, 1999)

4. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function such that \( |f(x) - f(y)| \geq |x - y| \) for every \( x, y \in \mathbb{R}. \) Show that the range of \( f \) is \( \mathbb{R}, \) i.e., for every \( c \in \mathbb{R} \) there is an \( x \) such that \( f(x) = c. \)

(De Souza & Silva, Berkeley Problems in Mathematics)

5. Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function, and define
\[ g(x) = f(x) \int_{0}^{x} f(t)dt. \]

Prove that if \( g \) is a nonincreasing function, then \( f(x) = 0 \) for every \( x. \)

(Romanian Olympiad 1978)

6. Let \( f : [0, 1] \to \mathbb{R} \) be a function with a continuous derivative, such that \( f(0) = 0 \) and \( 0 < f'(x) \leq 1 \) for every \( x. \) Show that
\[ \left( \int_{0}^{1} f(x)dx \right)^2 \geq \int_{0}^{1} (f(x))^3 dx. \]

(Putnam 1973)
7. Suppose \( f \) and \( g \) are \( n \)-times continuously differentiable functions in a neighborhood of a point \( a \), such that \( f(a) = g(a) \), \( f'(a) = g'(a) \), \ldots, \( f^{(n-1)}(a) = g^{(n-1)}(a) \), and \( f^{(n)}(a) \neq g^{(n)}(a) \). Evaluate

\[
\lim_{x \to a} \frac{e^{f(x)} - e^{g(x)}}{f(x) - g(x)}.
\]

(N. Georgescu-Roegen)

8. Let \( n \) be a positive integer, and \( f : [a, b] \to \mathbb{R} \) a continuous function, \( n \)-times differentiable on \((a, b)\). Prove that if the graph of \( f \) has \( n + 1 \) collinear points, then there is a point \( c \in (a, b) \) such that \( f^{(n)}(c) = 0 \).

(G. Sirețchi, *Mathematics Gazette, Bucharest*)

9. Suppose \( x_1, x_2, \ldots, x_n \in \mathbb{R} \). Find the real number(s) \( a \) that minimize the expression

\[
|a - x_1| + |a - x_2| + \cdots + |a - x_n|.
\]

(Andreescu & Gelca)

10. Prove that for every natural number \( n \geq 2 \) and every \( x \in [-1, 1] \),

\[
(1 + x)^n + (1 - x)^n \leq 2^n.
\]

(*Exercises and Problems in Algebra, Bucharest 1983*)