You are strongly urged to write up and hand in a careful and complete solution to (at least) one of these problems.

1. The horizontal line \( y = c \) intersects the curve \( y = 2x - 3x^3 \) in the first quadrant as in the figure. Find \( c \) so that the areas of the two shaded regions are equal. (Putnam, 1993)

2. A not uncommon calculus mistake is to believe that the product rule for derivatives says that \((fg)' = f'g'\). If \( f(x) = e^{x^2} \), determine, with proof, whether there exists an open interval \((a, b)\) and a nonzero function \( g \) defined on \((a, b)\) such that this wrong product rule is true for \( x \) in \((a, b)\). (Putnam 1988)

3. If \( n \) is a positive integer, prove for \( x > 0 \) that \[
\frac{x^n}{(x + 1)^{n+1}} \leq \frac{n^n}{(n + 1)^{n+1}}.
\]

4. (a) Assuming that temperature is a continuous function, show that at any given time on the earth’s equator there are two points directly opposite points that have the same temperature.
   
   (b) A rock climber starts to climb a mountain at 7:00 AM on Saturday and gets to the top at 5:00 PM. She camps on top and climbs back down on Sunday, starting at 7:00 AM. Show that at some time of day on Sunday she was at the same elevation as she was at that time on Saturday.

5. Suppose \( f \) and \( g \) are differentiable functions and for every \( x \), \( f'(x)g(x) \neq f(x)g'(x) \). Show that between every two zeros of \( f \) there is a zero of \( g \).

6. (a) Suppose that \( f(x) \) is continuous and \( f(x) \geq 0 \) on \([0, 1]\). Show that if \( \int_0^1 (x - 1)^2 f(x) \, dx = 0 \), then \( f(x) = 0 \) on \([0, 1]\).
   
   (b) Find all continuous functions \( f(x) \) on \([0, 1]\) such that \( f(x) \geq 0 \) and
   \[
   \int_0^1 f(x) \, dx = 1, \quad \int_0^1 xf(x) \, dx = \alpha, \quad \int_0^1 x^2 f(x) \, dx = \alpha^2
   \]
   
   where \( \alpha \) is a given real number. (Putnam, 1964)

7. Suppose \( f \) is a differentiable function on \([0, 1]\), \( f(0) = 0 \), and \( f'(x) \) is strictly increasing. Show that \( f(x)/x \) is strictly increasing.

8. Suppose \( f \) is a continuous function on \([0, 1]\), \( n \in Z^+ \), \( \int_0^1 x^k f(x) \, dx = 0 \) for \( k = 0, 1, \ldots, n - 1 \), and \( \int_0^1 x^n f(x) \, dx = 1 \). Show that there is a \( c \in [0, 1] \) such that \( |f(c)| > 2^n(n + 1) \).