Nonabelian Cohen-Lenstra Moments for the Quaternion Group

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1 Introduction and Background

1.1 Nonabelian Cohen-Lenstra heuristics

For a field k, let $Cl(k)$ denote the class group of the field. Write $Cl_p(k)$ for the Sylow p-subgroup of $Cl(k)$, called the p-part of the class group.

Conjecture 1 (Cohen-Lenstra [4]). *Fix an odd prime p and a finite abelian p-group A. The probability that $Cl(\mathbb{Q}(\sqrt{\pm d}))_p$ is isomorphic to A as $\pm d$ varies over negative (respectively positive) quadratic discriminants is*

$$Prob(Cl_p(\mathbb{Q}(\sqrt{-d})) \cong A) = c^-\frac{1}{\#\mathrm{Aut}(A)}$$
$$Prob(Cl_p(\mathbb{Q}(\sqrt{d})) \cong A) = c^+\frac{1}{\#A\#\mathrm{Aut}(A)}$$

For a pair of groups G and $G' \subset G \wr S_2$ Melanie Wood gave a generalization of Cohen-Lenstra heuristics by defining the **Cohen-Lenstra Moment** to be

$$E^\pm(G,G') = \lim_{X\to\infty} \frac{\displaystyle\sum_{0<\pm d<X} \#\{L/\mathbb{Q}(\sqrt{d}) \text{ unram.} : \mathrm{Gal}(L/\mathbb{Q}(\sqrt{d})) \cong G, \mathrm{Gal}(\widetilde{L}/\mathbb{Q}) \cong G'\}}{\displaystyle\sum_{0<\pm d<X} 1}$$

if the limit exists.

Conjecture 2 (Wood [9]). *For an admissible, good pair (G,G')*

$$E^-(G,G') = \frac{\#H_2(G',c)[2]}{\#\mathrm{Aut}_{G'}(G)} \qquad E^+(G,G') = \frac{\#H_2(G',c)[2]}{\#c\#\mathrm{Aut}'_{G}(G)}$$

and for an admissible, not good pair (G,G'), $E^\pm(G,G') = \infty$.

1.2 Quaternion Group

$$H_8 = \langle a,b,z : a^2 = b^2 = [a,b] = z, z^2 = [a,z] = [b,z] = 1\rangle$$

For any positive integer k, there exists a unique group $G'_k \subset H_8^k \wr S_2$ up to isomorphism with $[G'_k, H_8^k] = 2$ making (H_8^k, G'_k) an admissible pair

$$G'_k = H_8^k \rtimes S_2$$

This pair is not good.

Theorem 3 (A. (k=1), A.-Klys).

$$E^\pm(H_8^k, H_8^k \rtimes S_2) = \infty$$

Theorem 4 (Lemmermeyer [7]). *There exists an unramified extension $M/\mathbb{Q}(\sqrt{d})$ with $\mathrm{Gal}(M/\mathbb{Q}(\sqrt{d})) \cong H_8$ and $\mathrm{Gal}(M/\mathbb{Q}) \cong H_8 \rtimes S_2$ if and only if there is a factorization $d = d_1d_2d_3$ satisfying*

- *d_1, d_2, d_3 are coprime quadratic discriminants, at most one of which is negative.*
- $\left(\frac{d_1d_2}{p_3}\right) = \left(\frac{d_1d_3}{p_2}\right) = \left(\frac{d_2d_3}{p_1}\right) = 1$ *for primes $p_i \mid d_i$*

Moreover, for each factorization there are exactly $2^{\omega(d)-3}$ such extensions.

2 $E^\pm(H_8^k, H_8^k \rtimes S_2)$

2.1 $k=1$

Fix d_1 and d_2 and vary $d_3 = m$ to find the expected value of

$$a_{d_1,d_2,m} = \frac{1}{8}\sum_{d=d_1d_2m}\prod_{p\mid d}\left(1+\left(\frac{d_1d_2}{p}\right)\right)\left(1+\left(\frac{d_1m}{p}\right)\right)\left(1+\left(\frac{d_2m}{p}\right)\right)$$

Lemma 5.

$$\sum_{m \text{ sqf}} a_{d_1,d_2,m} m^{-s}$$

has a meromorphic continuation to $\mathbb{C}$ with no poles or zeroes in an open neighborhood of $Re(s) \geq 1$ except for a simple pole at $s=1$ with an explicit residue.

A Tauberian Theorem implies that

$$\sum_{d_1d_2m<X} a_{d_1,d_2,m} \sim \frac{c_{d_1,d_2}}{d_1d_2}X$$

which we can then sum over d_1, d_2 to show $E^\pm(H_8, H_8 \rtimes S_2) = \infty$.

2.2 Asymptotic Count implying $k \geq 1$

For an admissible pair (G,G') and a quadratic discriminant define

$$f_{G,G'}(d) = \#\{L/\mathbb{Q}(\sqrt{d}) \text{ unram.} : \mathrm{Gal}(L/\mathbb{Q}(\sqrt{d})) \cong G, \mathrm{Gal}(\widetilde{L}/\mathbb{Q}) \cong G'\}$$

Define the numerator of the fraction in $E^\pm(G,G')$ to be

$$N^\pm(G,G';X) = \sum_{0<\pm d<X} f_{G,G'}(d)$$

Theorem 6 (A.-Klys [3]).

$$N^-(H_8^k, H_8^k \rtimes S_2; X) \sim \frac{1}{4^k\#\mathrm{Aut}_\sigma(H_8^k)}\sum_{0<-d<X} 3^{k\omega(d)}$$
$$N^+(H_8^k, H_8^k \rtimes S_2; X) \sim \frac{1}{24^k\#\mathrm{Aut}_\sigma(H_8^k)}\sum_{0<d<X} 3^{k\omega(d)}$$

This implies $E^\pm(H_8^k, H_8^k \rtimes S_2) = \infty$ as a corollary. We also prove that normalized k^{th} moments are finite:

Theorem 7 (A.-Klys [3]).

$$E^-\left(\left(\frac{f_{H_8,H_8\rtimes S_2}(d)}{3^{\omega(d)}}\right)^k\right) = \frac{1}{32^k} \qquad E^+\left(\left(\frac{f_{H_8,H_8\rtimes S_2}(d)}{3^{\omega(d)}}\right)^k\right) = \frac{1}{192^k}$$

Use the same sieve as Fouvry-Klüners [5] did when finding the average value of the 4-rank of the class group of quadratic fields.

BIG IDEA: the main term only comes from those terms which cancel out completely by quadratic reciprocity, i.e. if $D_u, D_v \neq 1$ then $\Phi_k(u,v) = \Phi_k(v,u)$ so that the main term is

$$f(d)^k = \sum_{d=\prod D_u}\prod_{u,v}\left(\frac{D_u}{D_v}\right)^{\Phi_k(u,v)} \sim \sum_{d=\prod' D_u}\prod (-1)^{\Phi_k(u,v)\frac{D_u-1}{2}\frac{D_v-1}{2}}$$

3 Other Work in the Area

3.1 Other Results By Methods Related to the Ones in this Poster

- $(D_4, D_4 \times C_2)$ for D_4 the dihedral group of order 8. (A. [1])
- (G,G') for $[G':G] = 2$ and G' a central C_2 extension of C_2^m *not* containing a $C_2 \times D_4$ with $G \cap (C_2 \times D_4) = C_2 \times C_4$. (Klys [6])
- $(A, A \rtimes S_2)$ for A a finite abelian 2-group of exponent 8, and in a more recent preprint any finite abelian 2-group (Smith [8])

In a recent preprint [2], I generalize Lemmermeyer's key theorem classifying unramified H_8-extensions of $\mathbb{Q}(\sqrt{d})$ by certain factorizations $d = d_1d_2d_3$ to classify unramified G-extensions M/K of an abelian extension $K/\mathbb{Q}$ with $\mathrm{Gal}(M/K) = G'$ for pairs (G,G') such that $[G',G'] \leq Z(G')$.

This result can be used to do the case:

- (Upcoming) (G,G') for $[G':G] = 2$ and G' any central C_2^m extension of C_2^n

3.2 Future Directions

- Expected numbers of G-extensions M/K as K varies over abelian number fields with Galois group A with $\mathrm{Gal}(M/K) = G'$. My generalization of Lemmermeyer's conditions apply in this case for admissible pairs (G,G') with $[G',G'] \leq Z(G')$, suggesting that a similar sieve will work here too.
- Investigating the possibility that Smith's methods for $(A, A \rtimes S_2)$ for A an abelian 2-group can be applied to (G,G') for G any finite 2-group.
- What should we expect the main term of $N^\pm(G,G';X)$ to look like for not good pairs? This would be an expansion of Wood's conjecture.

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