Algebraic monodromy groups of l-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

Shiang Tang (under supervision of Stefan Patrikis)

University of Utah

Question

The classical Inverse Galois Problem asks whether every finite group can be realized as a Galois group over $\mathbb{Q}$. We ask an l-adic analogue:

For a connected reductive group G and a prime l, are there continuous homomorphisms

$$\rho : \Gamma_{\mathbb{Q}} \to G(\overline{\mathbb{Q}}_l)$$

with Zariski-dense images?

In this work, we give an answer to this question.

Main Theorem

Let G be a connected reductive algebraic group. There are continuous homomorphisms

$$\rho_l : \Gamma_{\mathbb{Q}} \to G(\overline{\mathbb{Q}}_l)$$

with Zariski-dense images for a positive density set of primes l if and only if the center of G has dimension at most one.

Why is it interesting?

The etale cohomology groups of algebraic varieties over $\mathbb{Q}$ are the natural source for l-adic representations of $\Gamma_{\mathbb{Q}}$. For example, the Tate module of a suitable n-dimensional abelian variety A over $\mathbb{Q}$ gives rise to a continuous representation

$$\rho : \Gamma_{\mathbb{Q}} \to \mathrm{GSp}_{2n}(\overline{\mathbb{Q}}_l)$$

with Zariski-dense image for all primes l. We say the classical group GSp_{2n} 'comes from geometry'. On the other hand, Zhiwei Yun and Stefan Patrikis have shown that most of the exceptional groups arise in a similar way. Is every connected simple algebraic group the Zariski closure of the image of some geometric representations of $\Gamma_{\mathbb{Q}}$ (in the sense of Fontaine-Mazur)? Probably not. In fact, assuming the Fontaine-Mazur and Langlands conjectures, it can be shown that there is *no* continuous homomorphism

$$\rho : \Gamma_{\mathbb{Q}} \to \mathrm{SL}_2(\overline{\mathbb{Q}}_l)$$

that is geometric (i.e., unramified almost everywhere and potentially semi-stable at l) and has Zariski-dense image! In contrast, our main theorem implies in particular that such a map *does* exist if we drop the geometric condition.

1. Our theorem gives an elegant classification of connected reductive groups that appear in continuous l-adic representations of $\Gamma_{\mathbb{Q}}$, while leaving the (seemingly very difficult) problem of determining which ones appear in geometric representations of $\Gamma_{\mathbb{Q}}$ for further research.
2. Our theorem contains the first sighting of $\mathrm{SL}_n, \mathrm{Sp}_{2n}, \mathrm{Spin}_n, E_6$ as any sort of arithmetic monodromy groups for $\Gamma_{\mathbb{Q}}$.

Method: Galois deformation theory

We start with a well-chosen mod l representation

$$\bar{\rho} : \Gamma_{\mathbb{Q}} \to G(\mathbb{F}_l)$$

and then use a variant of a method of Ravi Ramakrishna to deform it to $\mathbb{Z}_l$. Achieving this is a balancing act between two difficulties: the Inverse Galois Problem for finite groups of Lie types is difficult, so we want the residual image to be relatively 'small'; on the other hand, Ramakrishna's method requires the residual image to be 'big'.

Step 1: constructing the mod l representation

Take a maximal split torus T of $G(\mathbb{F}_l)$ and consider the following exact sequence of finite groups:

$$1 \to T \to N_G(T) \to W \to 1$$

where $W = N_G(T)/T$ is the Weyl group of G. We realize $N_G(T)$ (or a suitable subgroup of it) as a Galois group over $\mathbb{Q}$ satisfying certain ramification conditions, then define $\bar{\rho}$ to be the composite

$$\Gamma_{\mathbb{Q}} \twoheadrightarrow N_G(T) \subset G(\mathbb{F}_l)$$

Let $\bar{\rho}(\mathfrak{g})$ be the Lie algebra $\mathfrak{g}_{\mathbb{F}_l}$ equipped with a Galois action induced by the homomorphism $\Gamma_{\mathbb{Q}} \xrightarrow{\bar{\rho}} G(\mathbb{F}_l) \xrightarrow{Ad} \mathrm{GL}(\mathfrak{g}_{\mathbb{F}_l})$. This Galois module decomposes into $\mathfrak{t}$ (the Lie algebra of T) and a complement.

Step 2: deforming it to $\mathbb{Z}_l$

Now we want to find a *lift* of $\bar{\rho}$ to $\mathbb{Z}_l$, i.e., a continuous homomorphism $\rho : \Gamma_{\mathbb{Q}} \to G(\mathbb{Z}_l)$ whose reduction modulo l equals $\bar{\rho}$. We also want to ensure that the image of ρ is Zariski-dense in $G(\overline{\mathbb{Q}}_l)$. Suppose for $\bar{\rho}$ we have fixed a collection of local deformation conditions on a finite set of places S (containing the ramified and archimedean primes). Let $\mathcal{L}$ (resp. $\mathcal{L}^{\perp}$) be the Selmer system (resp. dual Selmer system) associated to the deformation conditions. By the Poitou-Tate exact sequence, if the dual Selmer group

$$H^1_{\mathcal{L}^{\perp}}(\Gamma_{\mathbb{Q},S}, \bar{\rho}(\mathfrak{g})(1))$$

vanishes, then there is a lift satisfying all the local deformation conditions.

Ravi Ramakrishna's idea is to impose finitely many additional local deformation conditions of 'Ramakrishna type' on a finite set of well-chosen places of $\mathbb{Q}$ disjoint from S in order to kill the dual Selmer group. Unfortunately, it does not work in our case. We overcome this by first observing that if the Selmer group

$$H^1_{\mathcal{L}}(\Gamma_{\mathbb{Q},S}, \mathfrak{t})$$

vanishes, then the dual Selmer group can be annihilated using Ramakrishna's method. However, given $\mathcal{L}$, $H^1_{\mathcal{L}}(\Gamma_{\mathbb{Q},S}, \mathfrak{t})$ may not vanish. Our second observation is, by making a variant of the Galois-cohomological arguments in Ramakrishna's method, we can first annihilate the '$\mathfrak{t}$-Selmer', then kill the full dual Selmer group $H^1_{\mathcal{L}^{\perp}}(\Gamma_{\mathbb{Q},S}, \bar{\rho}(\mathfrak{g})(1))$.