Extended Robba rings and the Fargues-Fontaine curve

Peter Wear University of California, San Diego

Email: pwear@ucsd.edu, preprint at https://arxiv.org/abs/1709.06221

Motivation: untilting perfectoid fields [3]

Let K be a nonarchimedean field, complete with respect to the multiplicative norm $|\cdot|$ and with residue characteristic p. Then K is *perfectoid* if the value group $|K^\times|$ is non-discrete and the p-th power Frobenius map on $\mathfrak{o}_K/p$ is surjective.

The p-adic completions of $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{Q}_p(\mu_{p^\infty})$ are examples of perfectoid fields. Every perfectoid field of characteristic p is perfect, a basic example of this is $\mathbb{F}_p((t^{1/p^\infty}))$ - the completion of the perfection of $\mathbb{F}_p((t))$.

Given any perfectoid field K, define the tilt $K^\flat$ as $\lim\limits_{x\mapsto x^p} K$. This is a perfectoid field of characteristic p.

All three of the above examples tilt to $\mathbb{F}_p((t^{1/p^\infty}))$.

For any perfectoid field K, the absolute Galois groups of K and $K^\flat$ are isomorphic.

Let L be a perfectoid field of characteristic p and K a characteristic 0 *untilt* of L, so $K^\flat \cong L$. There is a surjection $\Theta : W(\mathfrak{o}_L)[\frac{1}{p}] \to K$ with kernel generated by an element ξ which is the Witt vector analogue of a linear polynomial. In this way, untilts of L correspond to maximal ideals of $W(\mathfrak{o}_L)[\frac{1}{p}]$.

The Fargues-Fontaine curve is a geometric object acting as a "moduli space of untilts". We study the Fargues-Fontaine curve via the rings it is built out of: the extended Robba rings.

The adic Fargues-Fontaine curve [3]

Fix a prime p and a power q of p. Let L be a perfectoid field containing $\mathbb{F}_q$ with pseudo-uniformizer π, let E be a complete discretely valued field with residue field containing $\mathbb{F}_q$ and uniformizer ϖ. Let $\mathcal{Y}_{L,E} = \mathrm{Spa}(W(\mathfrak{o}_L)\otimes_{W(\mathbb{F}_q)}\mathfrak{o}_E)\setminus\{|\varpi[\pi]| = 0\}$. Frobenius on $\mathfrak{o}_L$ induces a properly discontinuous automorphism ϕ on $\mathcal{Y}_{L,E}$.

The adic Fargues-Fontaine curve is $\mathcal{X}_{L,E} := \mathcal{Y}_{L,E}/\phi$.

The Fargues-Fontaine curve geometrizes many important concepts from p-adic Hodge theory. The closed points correspond to the untilts of L containing E. Vector bundles over the Fargues-Fontaine curve relate to (ϕ,Γ)-modules, which are used to study p-adic Galois representations.

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An analogy: Tate's elliptic curve [1]

Let k be a nonarchimedean field with norm $|\cdot|$, choose an element $q \in k^\times$ with $|q| < 1$. The quotient $E_q = \mathbb{G}_{m,k}/(q)$ makes sense in the category of rigid analytic spaces, it is the analytification of an elliptic curve. The rigid space $\mathbb{G}_{m,k}$ can be covered by annuli $A(r,s)$ centered at the origin for intervals $[r,s] \subset (0,\infty)$ - these are affinoid rigid spaces corresponding to the affinoid algebras $k\langle r/X, X/s\rangle \cong k\langle T_1,T_2\rangle/(T_1T_2 - r/s)$. When $r/s < |q|$, the quotient $\mathbb{G}_{m,k}\to E_q$ maps $A(r,s)$ isomorphically to its image. We can learn about E_q by studying the affinoid algebras that it is built out of, which we can understand by studying the Tate algebra $k\langle T_1,T_2\rangle$.

The extended Robba rings [2]

The adic space $\mathcal{Y}_{L,E}$ is covered by rational subspaces $\mathcal{Y}^I_{L,E}$ corresponding to intervals $I = [a,b] \subset (0,\infty)$ defined by the inequalities $\{[\pi^b] \le |\varpi| \le [\pi^a]\}$. These subspaces are affinoid adic spaces corresponding to the extended Robba rings $B^I_{L,E}$.

For sufficiently small intervals I, the quotient $\mathcal{Y}_{L,E}\to\mathcal{X}_{L,E}$ maps $\mathcal{Y}^I_{L,E}$ isomorphically to its image. We can learn about the Fargues-Fontaine curve by studying the extended Robba rings $B^I_{L,E}$ and the somewhat simpler rings $A^r_{L,E}$ which are analogues of the Tate algebra.

The ring $A^r_{L,E}$ is the subring of $W(L)\otimes_{W(\mathbb{F}_q)}\mathfrak{o}_E$ consisting of elements $\sum_{i=0}^{\infty}\varpi^i[\overline{x}_i]$ satisfying a convergence condition involving r. It is complete for a multiplicative norm and has a degree function analogous to the Gauss norm and degree function of the Tate algebra $k\langle T\rangle$.

In particular, $A^r_{L,E}$ is a Euclidean domain for its degree function and is therefore a principal ideal domain.

This is a crucial input to the proof that $A^r_{L,E}$ is strongly noetherian, that is that for any $\rho_1,\dots,\rho_n > 0$ the ring $A^r_{L,E}\{T_1/\rho_1,\dots,T_n/\rho_n\}$ is noetherian. This result extends to the rings $B^I_{L,E}$ and étale extensions C of these rings as they can all be written as quotients of some $A^r_{L,E}\{T_1/\rho_1,\dots,T_n/\rho_n\}$. This implies that the structure presheaf on $\mathcal{X}_{L,E}$ is a sheaf, and therefore that $\mathcal{X}_{L,E}$ is an adic space.

All of our results are obtained in a similar way - we work directly with the ring $A^r_{L,E}$ and extend to the rings $B^I_{L,E}$ and C by taking quotients of weighted Tate algebras over $A^r_{L,E}$.

Results

Throughout this section, let $(B^I_{L,E}, B^{I,+}_{L,E}) \to (C, C^+)$ be a morphism of adic Banach rings which is étale in the sense of Huber. All results are already new in the case $C = B^I_{L,E}$

We give a classification of the points of $\mathrm{Spa}(C,C^+)$ into types 1-5, analogously to the classification of the points of $\mathrm{Spa}(k\langle t\rangle,\mathfrak{o}_k\langle t\rangle)$. Types 1-4 correspond to the points of the Berkovich spectrum $\mathcal{M}(C)$, type 1 points correspond to the points of $\mathrm{MaxSpec}(C)$.

To show this, we first extend Kedlaya's classification of $\mathcal{M}(W(R))$ into types 1-4 by checking that the proofs still work for generalized Witt vectors. We then show that all higher rank points must be clustered around points of type 2, the argument is analogous to the equivalent statement for the adic unit disc.

We show that a finite collection of rational subspaces of $\mathrm{Spa}(C,C^\circ)$ forms a covering if and only if their intersections with $\mathrm{MaxSpec}(C)$ do. We see that a rational subspace of $\mathrm{Spa}(C,C^\circ)$ is determined by its intersection with $\mathcal{M}(C)$, then show that for any rational localization $\mathrm{Spa}(C,C^\circ)\to\mathrm{Spa}(D,D^+)$ we have $D^+ = D^\circ$.

For these proofs, our classification gives us a good understanding of how the higher rank points behave. Rational localizations of adic spaces are defined by inequalities, and we can show that inequalities satisfied by type-5 valuations are also satisfied by nearby type-2 valuations.

We show that for a "dense set of rings C" and a "dense set of $\rho_1,\dots,\rho_n > 0$", the ring $C\{T_1/\rho_1,\dots,T_n/\rho_n\}$ is regular and excellent.

The key step in the proof of regularity is a version of the Nullstellensatz: we show that every maximal ideal $\mathfrak{m}$ of $A^r_{L,E}\{T_1/\rho_1,\dots,T_n/\rho_n\}$ restricts to a maximal ideal of $A^r_{L,E}$. This lemma forces the restrictions on C and the ρ_i.

To show excellence in characteristic 0, we first show that $A^r_{L,E}\{T_1/\rho_1,\dots,T_n/\rho_n\}\otimes_{A^r_{L,E}}\mathrm{Frac}(A^r_{L,E})$ is excellent because it satisfies the "weak Jacobian condition". Roughly, this is because it is regular of dimension n and there are n independent derivations $\frac{\partial}{\partial T_i}$.

References

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