Curves of Genus 2 with Elliptic Differentials and Related Hurwitz Spaces

Gerhard Frey
IEM
University of Duisburg-Essen

SCNTD
October 14, 2006

This is joint work with

E. Kani

and, partly, with

C.Diem

and

H. Völklein

Contents

K is a field of characteristic $p \geq 0$, $p \neq 2$
of finite type over its prime field K_0
with absolute Galois group G_K.
$n \in \mathbb{N}$ is always assumed to be prime to p.
The representation induced by the action of G_K on the n-torsion points $E[n]$ of an elliptic curve E over K is denoted by $\rho_{E,n}$

1 Isomorphic Torsion Structures

Conjecture 1.1 ***Darmon:***
There is a number $n_0 = n_0(K)$ such that for all E, E' over K and all $n \geq n_0$ we have

$$\rho_{E,n} \cong \rho_{E',n}$$

iff

$$E \text{ is } K\text{-isogenous to } E'.$$

Kani:
Maybe there are finitely many exceptions, and: for $K = \mathbb{Q}$ the bound $n_0 = 23$ suffices.

Much weaker is

Conjecture 1.2 *For all elliptic curves E_0 over K there is a number $n_0 = n_0(K, E_0)$ such that for all E defined over K we get:*
For $n \geq n_0$ it is equivalent

1. $\rho_{E,n} \cong \rho_{E_0,n}$
2. *E is isogenous to E_0.*

For global fields (or more generally, fields with divisor theory satisfying finiteness conditions) this conjecture would follow from the *height conjecture* for elliptic curves.
Since the height conjecture is true over function fields the hard case is that K is a number field.

Assume that $K = \mathbb{Q}$.

- The height conjecture is equivalent with the ABC-conjecture *and* with the *degree conjecture for modular parameterizations:* There exist $c, d \in R_{>0}$ such that for minimal

$$\varphi : X_0(N_E) \longrightarrow E$$

$$\log(\deg(\varphi)) \leq c \log(N_E) + d$$

- equivalences between Galois representations on torsion points of elliptic curves correspond to congruences of cusp forms.
- The *Asymptotic Fermat Conjecture* is equivalent with Conjecture 1.2 for even n .

2 Hurwitz Spaces

2.1 A “special” Hurwitz surface

Look at covers

$$\varphi : \mathbb{P}^1_K \to \mathbb{P}^1_K$$

of (for simplicity) *odd* degree n which are primitive (i.e. has no proper intermediate subcovers) and which have the following ramification behavior:

(∗) φ is ramified in 5 points $P_1, \ldots, P_5$ with ramification order at most 2, and the ramification cycle corresponding to P_5 in the Galois closure of the cover is a transposition.

This condition clearly implies that P_5 has exactly one ramified extension Q_5 in the cover and that P_5, $Q_5 \in \mathbb{P}^1(K)$.

Let r_i denote the number of *unramified* extensions of P_i, for $1 \leq i \leq 4$. Since

$$-2 = -2n+1+\sum_{1\leq i\leq 4}\frac{(n-r_i)}{2} = 1-\sum_{1\leq i\leq 4}\frac{r_i}{2}.$$

we get $\sum r_i = 6$.
Since n is odd, r_i is odd and so there are

- P_1, P_2, P_3 with $r_i = 1$
- P_4 with $r_4 = 3$

It follows that P_4, P_5, $P_1+P_2+P_3$ and the discriminant divisor

$$\mathrm{disc}(\varphi) = P_1 + \ldots + P_5$$

are K-rational.

The Galois closure $\overline{\varphi}$ of a cover φ : $\mathbb{P}^1 \to \mathbb{P}^1$ satisfying $(*)$ has Galois group S_n and has a fixed ramification cycle type $\mathcal{C}$. So we get a moduli functor.
Call two covers

$$\varphi_1, \varphi_2 : \mathbb{P}^1 \to \mathbb{P}^1$$

to be *weakly equivalent* if there exist $\alpha_i \in \mathrm{Aut}(\mathbb{P}^1)$ such that

$$\varphi_1 \alpha_1 = \alpha_2 \varphi_2.$$

The resulting quotient moduli functor is *coarsely* represented by the quotient space $H_n^* = H^{in}(S_n, \mathcal{C})/Aut(\mathbb{P}^1)$. (**Fried, Völklein et al**).
It is a (non-complete) irreducible surface.

Remark 2.1 (a) Since H_n^* is only a coarse moduli space for the quotient functor, it is more difficult to characterize the K-rational points of H_n^*.
(b) The basic existence result (Riemann) yields that there always exist such covers (over algebraically closed ground fields of characteristic 0) for arbitrary points $P_1, \ldots, P_5$.
But it seems more difficult to determine explicitly how many such covers exist and to decide rationality questions over arbitrary ground fields . (Indeed, the underlying ramification cycle structure is highly non-rigid!)
Moreover, the constructions shed little light on the actual structure of the Hurwitz spaces.

2.2 Covers of Elliptic Curves by Curves of Genus 2

Why is H_n^* interesting? It is characterized by

- covers of $\mathbb{P}^1$ of odd degree by itself
- with 5 ramification points
- one ramification cycle is a transposition.

Trivial observation

$P_1, \cdots, P_4$ determine an elliptic curve E/K unique up to a quadratic twist. For x of $\mathbb{P}^1_K$ with $(x)_\infty = P_4$ take

$E : y^2 = f_3(x)$ with $(f_3)_0 = P_1 + P_2 + P_3$.

Let $C = E \times_{\mathbb{P}^1} \mathbb{P}^1$ be the normalization of the fibre product of E and $\mathbb{P}^1$ over $\mathbb{P}^1$ (with respect to the morphisms $\pi : E \longrightarrow \mathbb{P}^1$ and φ).
C is an irreducible curve of genus 2 satisfying a hyperelliptic equation of the form

$$Y^2 = f_3(X) \cdot g_3(X)$$

with g_3 a polynomial of degree 3 corresponding to the 3 unramified extensions of P_4 in the cover φ.

C comes together with morphisms

$$f : C \longrightarrow E$$

and

$$\pi' : C \longrightarrow \mathbb{P}^1.$$

$$\begin{array}{ccc}
 & & C \\
 & \overset{\pi'}{\swarrow} & \\
\mathbb{P}^1 & & \downarrow f \\
\varphi \downarrow & & E \\
 & \overset{\pi}{\swarrow} & \\
\mathbb{P}^1 & &
\end{array}$$

Properties of f

1. The morphism $f : C \longrightarrow E$ is minimal.
2. For the *Weierstraß divisor* $W_C = W_1 + \ldots + W_6$ we have
$$f_* W_C = 3 \cdot 0_E + P_1' + P_2' + P_3'. \quad (1)$$
3. $$f \circ \omega_C = [-1]_E \circ f, \quad (2)$$
where ω_C denotes the hyperelliptic involution of C.

Definition 2.2 *A cover* $f : C \to E$ *with properties* 1, 2 *and* 3 *is called* normalized.

By the Riemann-Hurwitz genus formula we see that the different of f has to be a divisor of degree 2.

In our situation it is easily identified: The point P_5 has two distinct extensions P and $P' = [-1]_E P$ to E, and there is exactly one point Q resp. $Q' = \omega Q$ over P resp. P' which is ramified of order 2. Hence the discriminant divisor of f is equal to $\pi^*(P_5)$.

Conversely, assume that

$$f_0 : C \longrightarrow E$$

is a minimal cover of an elliptic curve E by a curve of genus 2 of odd degree n defined over K.

Lemma 2.3 *There is a unique translation $\tau : E \longrightarrow E$ such that*

$$f = \tau \circ f_0$$

is normalized.

f factors over the hyperelliptic cover $\pi' : C \longrightarrow C/\langle\omega\rangle = \mathbb{P}^1$. and induces a primitive cover $\varphi : C/\langle\omega\rangle = \mathbb{P}^1 \longrightarrow E/\langle -id_E\rangle = \mathbb{P}^1$ of degree n such that $\varphi \circ \pi' = \pi \circ f$.

Let $P_5 \in \mathbb{P}^1(K)$ be such that $\mathrm{Disc(f)} = \pi^*(\mathrm{P}_5)$, $P_1 + P_2 + P_3 + P_4 = \pi_*(E[2])$. Then

$$\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

defines a point in H_n^* iff $\mathrm{Disc(f)}$ is reduced, i.e. if and only $P_5 \notin \{P_1, \dots, P_4\}$ (generic case).

Consider the moduli problem $\mathcal{H}_n$ which classifies isomorphism classes of pairs (C, f) where C is a curve of genus 2 and f is a *normalized* covering map from C to an elliptic curve E of degree n.
Let $\mathcal{H}'_n$ denote the subproblem which classifies the covers for which Disc(f) is reduced.
We shall find surfaces H_n and H'_n which represent these problems coarsely.

3 Moduli Functors

3.1 The “Basic Construction”

Observation: A normalized cover

$$f : C \to E$$

induces a map

$$f_* : J_C \to E$$

whose kernel is an elliptic curve $E' \subset J_C$ intersecting $f^*(E)$ exactly in $E'[n]$. Hence it gives rise to a triple

$$(E, E', \alpha_n : E[n] \to E'[n])$$

over K. J_C is (as p.p. variety)

$$(E \times E')/graph(\alpha_n)$$

and α_n is anti-isometric with respect to the Weil pairing.

Conversely: To a triple

$$(E, E', \alpha_n)$$

satisfying the conditions from above the abelian variety

$$(E \times E')/graph(\alpha_n)$$

has a principal polarization C (ie. a curve of genus 2) and a cover map

$$f : C \longrightarrow E$$

of degree n which is normalized *iff* C is irreducible. It follows that the moduli problem $\{(E, E', \alpha_n)\}$ has $\mathcal{H}_n$ and $\mathcal{H}'_n$ as subproblems.

3.2 Twisted Modular Curves

Now fix E or, in the language of Hurwitz spaces, the points $P_1, \ldots, P_4$. So P_5 is varying. As result we get a curve, and, again in the language of covers of E, the "parameter" is the discriminant divisor.

More precisely:
For any extension field L of K, let $E_L = E \otimes L$ denote the elliptic curve E lifted to L. We now consider the set

$$\mathrm{Cov_{E/K,n}(L)} :=$$

$$\{f : C \to E_L : f \text{ normalized,defined over } L,$$
$$\deg(f) = n\}/Aut E_L$$

The assignment $L \mapsto \mathrm{Cov}_{\mathrm{E/K,n}}(\mathrm{L})$ can be extended in a natural way to a functor $\mathcal{H}_{E/K,n} : \underline{Sch}_{/K} \to \underline{Sets}$.

Theorem 3.1 ***(Kani)***
If $n \geq 3$, then the functor $\mathcal{H}_{E/K,N}$ is finely represented by a smooth, affine and geometrically connected curve $H_{E/K,n}/K$ with the property that $H_{E/K,N} \otimes K_s$ is an open subset of the modular curve $X(N)_{/K_s}$.

The proof of this theorem uses the basic construction.

Remark 3.2 The fact that the curve $H := H_{E/K,n}$ *finely* represents the functor $\mathcal{H}_{E/K,n}$ means that there exists a *universal* normalized genus 2 cover

$$f_H : \mathcal{C}_H \to E \times H$$

of degree n with the property that every normalized genus 2 cover $f : C \to E \times S$ of degree n (where S is any K-scheme) is obtained uniquely from f_H by base-change. In particular, the set $\mathrm{Cov_{E/K,n}(K)}$ of covers can be identified with the set of fibres $f_x := (f_H)_x : \mathcal{C}_x \to E_x = E$ of f_H, where $x \in H(K)$.

3.3 Varying the Base Curve

We want to represent the moduli functors $\mathcal{H}_n$ and $\mathcal{H}'_n$.
We can do this already for “fibers” (fixed choice of $P_1, \ldots, P_4$).
So we have to glue together.
But we can expect only coarse moduli schemes.
So a discussion of K-rational points is not enough, we shall have to study the associated moduli functors and/or *stacks* in more detail.
The appropriate frame is given by the concept of
moduli problems for elliptic curves introduced by *Katz-Mazur.*

It uses the *moduli stack* $\underline{Ell}_{/R}$ of all elliptic curves over a ring R where the objects are relative elliptic curves over R-schemes S.

Moduli problems $\mathcal{P}$ give rise to contravariant functors

$$\widetilde{\mathcal{P}} : \underline{Sch}_{/R} \to \underline{Sets}$$

which classify isomorphism classes of $\mathcal{P}$-structures.

Example 3.3 *We take $R = \mathbb{Z}[1/2n]$ and $\mathcal{P}$ the functor $\mathcal{Z}_{n,-1}$ which associates to E/S the set of elliptic curves defined over S with isomorphic n-torsion structure with isotropic graph.*
We know already that this functor is finely representable for fixed E and get a moduli space *$Z_{E,n,-1}$ defined over R. Take j transcendental over R and for E_j a curve with invariant j. Define $Z_{E_j,n,-1} =: Z_{n,-1}$.*

Theorem 3.4 (F-Kani) *The moduli problems $\mathcal{H}_n$ and $\mathcal{H}'_n$ are coarsely representable by open subschemes H_n and H'_n of $Z_{n,-1}$ which is normal and affine and of relative dimension 1 over $Spec(\mathbb{Z}[1/2n][j]) = M(\Gamma[1])$.*

3.4 Diagonal Surfaces

For simplicity take $R = \mathbb{Z}[\frac{1}{2n}, \zeta_n]$.
Take the modular affine curve

$$X(n)'_{/\mathbb{Z}[\frac{1}{2n},\zeta_n]}$$

with the action of

$$G_n = \mathrm{SL}_2(\mathbb{Z}/\mathrm{n}\mathbb{Z})/\{\pm 1\}.$$

$G_n \times G_n$ acts on the product

$$Y_n = X(n)'_{/\mathbb{Z}[\frac{1}{2n},\zeta_n]} \times_{\mathbb{Z}[\frac{1}{2n},\zeta_n]} X(n)'_{/\mathbb{Z}[\frac{1}{2n},\zeta_n]}.$$

Definition 3.5 *Define*

$$\sigma_{-1} := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Denote by Δ_{-1} *the graph of the conjugation by* σ_{-1}.
The quotient

$$Z_{n,-1} = Y_n/\Delta_{-1}$$

is the modular diagonal quotient surface *of type* $(n, -1)$.

Proposition 3.6 *The modular diagonal quotient surface* $Z_{n,-1}$ *is the coarse moduli space of the moduli problem* $\mathcal{Z}_{n,-1}$ *and hence the coarse moduli spaces* H_n *and* H'_n *are open subsets of* $Z_{n,-1}$.

3.5 Back to the Hurwitz World

We analyze the space H_n^* by the same techniques (stacks) as above replacing $\underline{Ell}/R$ by the stack of curves of genus 0 over R,$\underline{M_0}$ where R is a ring in which $n!$ is invertible.

Let $X/S \in ob((\underline{M_0})_{/R})$ be a relative genus 0 curve, and consider the set

$$\mathrm{Cov}^*_{\mathrm{X/S,n}} :=$$

$$\{\varphi : Y \to X : \varphi \text{ finite cover of degree } n$$
$$\text{whose fibres } \varphi_t \text{ satisfy } (*)\}/ \simeq_X .$$

Then the rule $\mathcal{H}_n^*(X/S) = \mathrm{Cov}^*_{\mathrm{X/S,n}}$ defines a moduli problem on $(\underline{M_0})_{/R}$, i.e. a functor $\mathcal{H}_n^* : (\underline{M_0})_{/R} \to \text{Sets}.$

Theorem 3.7 *The moduli functor* $\mathcal{H}_n^*$ *has a coarse moduli space.* *The rule*

$$(C \xrightarrow{f} E) \mapsto (C/\langle\omega_C\rangle \to E/\langle[-1]\rangle)$$

defines a functor

$$q = q_n : \underline{\mathcal{H}}'_n \to \underline{\mathcal{H}}^*_n,$$

and the induced map

$$M(q) : H'_n = M(\mathcal{H}'_n) \ \to \ H^*_n = M(\mathcal{H}^*_n)$$

on the coarse moduli schemes is surjective and radical. Thus, if $R = K$ *is a field of characteristic* 0*, then* $M(q)$ *is an isomorphism and hence*

$$H'_n \simeq H^*_n$$

is an irreducible, normal affine surface.

4 Compactification

As the constructions of the previous section show, the moduli spaces H_n' and H_n^* are not compact. It is thus of interest to construct natural compactifications of these spaces and to investigate whether or not the boundary components have a modular interpretation in terms of covers of curves.

Since H_n' was constructed as an open subset of the affine surface $Z_{n,-1}$, the natural compactification $\overline{Z}_{n,-1}$ of $Z_{n,-1}$ also serves as a compactification of H_n'.

A modular interpretation can be obtained by studying the degeneration of the canonical compactification of the universal cover $C \longrightarrow E_H$ over $H = H_{E/K,n}$ for each fixed elliptic curve E/K. On the other hand, the theory of **Wewers et al** gives a recipe for an (abstract) compactification of $H^{in}(S_n, \mathbb{C})$ in terms of covers, and this also gives a compactification of $H_n^* = H^{in}(S_n, \mathbb{C})/\mathrm{Aut}(\mathbb{P}^1)$.

Thus, in both interpretations the boundary curves of the Hurwitz spaces correspond to interesting degenerations of covers.

This leads to a nice picture about the interplay of geometry with group theory. We get a complete dictionary between degenerations with respect to the modular interpretation by covers of curves of genus 2 to elliptic curves (here level-n-structures and geometry give a complete classification), and on the other side with respect of degenerations of ramification points and ramification cycles (with a complete classification given by group theory (joint work with **H. Völklein**).We shall give only a few hints.

4.1 The boundary curves of H'_n

The boundary curves naturally split into three types induced by the inclusions

$$H'_n \subset H_n \subset Z_{n,-1} \subset \overline{Z}_{n,-1}.$$

Type 1: $\partial_1 := H_n \setminus H'_n$

By the modular description of H_n mentioned in subsection 2.2, the points of ∂_1 classify genus 2 covers $f : C \to E$ whose discriminant is not reduced, i.e. Disc(f) = 2P. We get $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ in which $P_5 = P_i$, for $i = 1, \ldots, 4$. Hence we can understand this degeneration as a coalescence process. Various possible ramification types can occur (predicted by group theory) and so ∂_1 decomposes further into several components consisting of curves with growing genus.

Type 2: $\partial_2 := Z_{n,-1} \setminus H_n$
It is immediate from the "basic construction" that the points in ∂_2 correspond to singular (stable) genus 2 curves whose Jacobian is smooth, i.e. to curves C that are the union of two elliptic curves meeting in a single point. Again this means that P_5 moves to one of the points $P_1, \ldots, P_4$.

Type 3: $\partial_3 := \overline{Z}_{n,-1} \setminus Z_{n,-1}$

$$\partial_3 = C_{\infty,1} \cup C_{\infty,2}$$

is the union of two irreducible curves $C_{\infty,i} \simeq X_1(n)$. The two cuspidal curves $C_{\infty,i}$ correspond to two different (sub)types of degenerations of covers $f : C \to E$. In the first type the curve C degenerates to a singular curve of genus 2 (whose normalization is an elliptic curve), and in the second subtype the curve E degenerates to singular curve of arithmetic genus 1 (whose normalization is therefore $\mathbb{P}^1$).
In terms of Hurwitz spaces this means that points in $P_1, \ldots, P_4$ coalesce.

4.2 The Rigidity Number

We shall give a finer application using geometry to get information about group theory and compute the "number of non-rigidity" of H_n^*.

Theorem 4.1 ***(Kani)*** *Let $n > 1$ be an odd integer, and suppose that K is an algebraically closed field. If either char$(K) = 0$ or char$(K) > n$ then the number of covers of degree n ramified in P is*

$$\frac{1}{12}(n-1)\#SL_2(\mathbb{Z}/n\mathbb{Z}).$$

Remark 4.2 This number is a measure of non-rigidity of the ramification type which defines H_n^*. Although this number can be directly defined via group theory, it seems very difficult, if not impossible, to compute this number in this way. The basic trick of the proof here is to compute instead a related number in the case of "degenerate covers".

5 Rational Points

5.1 The Conjectures Restated

We come back to the conjectures stated in the first section.
We are looking for triples

$$(E, E', \alpha_n : E[n] \to E'[n])$$

defined over K.
We restrict ourselves for simplicity to the case that the determinant of α_n is $= -1$.
Conjecture 1.2 predicts that certain *curves* on $Z_{n,-1}$, namely *twisted modular curves*, have, for n large enough, only obvious points.

Conjecture 1.1 predicts that, for n large enough, *points* in $Z_{n,-1}(K)$ correspond to pairs of elliptic curves (E, E') which are isogenous.
This is supported by the following result of Hermann, Kani and Schanz.

Proposition 5.1 *For* $n \leq 10$ *the surface* $Z_{n,-1}$ *is rational or a* $K-3$*-surface or an elliptic surface.*
For $n \geq 11$ *it is a surface of general type.*

Recall *Lang's conjecture* which predicts that for such surfaces the K-rational points are concentrated on curves of genus ≤ 1.

Hence it is interesting to find "obvious" curves on $Z_{n,-1}$ and then to prove that there are no other curves of low genus on this surface.

Unfortunately, we do not see theoretical methods to come nearer to the second part of this task.

Since the surfaces to be studied are given in a most explicit way one could think of doing computational experiments, i e for $n = 23$, the smallest interesting example.

5.2 Points related to Isogenies

The conjectures stated in Subsection5.1 motivate to *assume* in the following that

$$\eta : E \to E'$$

is a cyclic $K-$isogeny of minimal degree denoted by l_0.

To avoid trivial cases *we always assume that $l_0 > 1$ and η is separable.*

Denote by α_n the restriction of η to $E[n]$.

Of course, there may be other G_K-isomorphisms between $E[n]$ and $E'[n]$. We call triples $(E, E', z \cdot \alpha_n)$ "generic" because of

Lemma 5.2 *Assume that the centralizer of G_K in $Aut(E_n)$ is $\mathbb{Z} \cdot id_{E[n]}$ and that n is prime to l_0. Then every G_K-isomorphism between $E[n]$ and $E'[n]$ is of the form $z \cdot \alpha_n$ with $z \in \mathbb{Z}$ prime to l_o. In particular, this is the case if E has no complex multiplication and n is large enough (depending on E and K).*

An easy observation is

Proposition 5.3 *For all n prime to l_0 and all $z \in \mathbb{Z}$ prime to n the abelian variety $J_n := (E \times E')/graph(z \cdot \alpha_n)$ is isomorphic to $E \times E'$.*

In order to get points on H_n via the "basic construction" we need two additional properties:

Firstly the graph of $z \cdot \alpha_n$ has to be isotropic with respect to the Weil pairing and secondly one has to verify that the resulting curve C is irreducible.

Proposition 5.4 *Assume that l_0 is square free and that n is aprime. Assume in addition that E has no complex multiplication. Then there is an element $z \in \mathbb{Z}$ such that $z \cdot \alpha_n$ induces a covering*

$$C \to E$$

of degree n iff $n = 2$ or n is split into two non principal prime ideals in $\mathbb{Q}(\sqrt{-l_0})$.

5.2.1 Universal construction

Take l_0 squarefree such that $\mathbb{Q}(\sqrt{-l_0})$ has class number > 1.
Let $F_0 := K_0(j, j_0)$ be the function field of $X_0(l_0)/K_0$.
Let E_j be an elliptic curve with invariant j defined over F_0.
Then there is a curve C of genus 2 defined over F_0 and infinitely many numbers n such that C is covering E_j of degree n. If l_0 is odd we get curves birationally equivalent to $X_0(l_0)$ on the Hurwitz spaces H'_n resp. H^*_n.
For instance, take $l_0 = 5$ Then $X_0(l_0)$ has genus 0 and so for infinitely many n we get a rational curve in H'_n and H^*_n.

5.3 Degree of covers

We have seen in Subsection 5.2.1 that under appropriate conditions there are fields of finite type K and pairs (E, E') of isogenous elliptic curves which are $K-$rationally covered by curves C of genus 2 with Jacobian variety $E \times E'$. In fact, there are *infinitely* many such covers f_n (if there is one) with the same cover curve C.

It is an interesting task to determine the degrees of these covers. It leads to questions about representation of numbers by quadratic forms.

We do not go into details in general but restrict ourselves to a special case.

Let $\widehat{\eta}$ be dual to the isogeny $\eta : E \to E'$ of degree l_0.
Let $\pi : C \to E$ and $\pi' : C \to E'$ be covers of degree n with $J_C = E \times E'$ and

$$\pi_* \circ \pi'^* = 0_E \text{ and } \pi'_* \circ \pi^* = 0_{E'}.$$

.

Lemma 5.5 ***(Diem-F)***
For $a, b \in \mathbb{Z}$

$$a\pi + b\widehat{\eta}\pi' : C \longrightarrow E$$

has

$$\deg(a\pi + b\widehat{\eta}\pi') = na^2 + nl_0b^2.$$

If $a = 1$ and $b \equiv 0 \mod n$ the cover is minimal.

5.4 Ramification points

We continue to assume that E and E' are isogenous curves with a $K-$rational isogeny η of minimal odd degree l_0 and that the $j-$invariant of E is not contained in the prime field K_0 of K and that $f : C \to E$ is f is induced by the universal construction described in Subsection 5.2.1.It follows that there are two ramification points of f on C and two branch points of f on E corresponding to a point $P_5(f) \in \mathbb{P}^1(K)$ with unique ramified extension $Q_5(f) \in \mathbb{P}^1(K)$.
We have seen that there are infinitely many covers $f_n : C \to E$ of different degrees n.

Question:
How do P_5 resp. Q_5 move when we change f (with fixed C).

Proposition 5.6 $Q_5(f)$ *corresponds to an ideal in* $K_0[X]$ *of degree bounded by a number depending only on* E*.*

The proof is not trivial; it uses results from from arithmetical geometry of fibred surfaces.

This result could indicate finiteness properties of ramification resp. branch points of covers.
We remark that finiteness of $\{P_5(f_n)\}$ yields finiteness of $\{Q_5(f_n)\}$ since $K-$rationality of P_5 implies K-rationality of the corresponding ramification point on C.

A closer look (F-$Diem$) reveals that the worlds in characteristic 0 and characteristic > 0 behave differently.

Proposition 5.7 *Let K be a field of characteristic 0.*
Let c_1 and c_2 be minimal covers from C to E with $c_1 \neq \pm c_2$. Then the ramification loci of c_1 and c_2 are disjoint.
So, if $\{f_n; n \in I\}$ is an infinite set of minimal covers from C to E then the set of ramification loci and *the set of branch loci of these covers are infinite.*

Proposition 5.8 *Assume that char(K)= $p > 0$. Then all ramification points for all minimal covers from C to E lie in a finite set of order* $\leq p^{\dim_{\mathbb{Z}} Hom_K(J_C,E)}$, *and there is a cover*

$$C \to E$$

of degree 2 such that there are infinitely many covers

$$f_n : C \to E$$

defined over K with $P_5(f) = P_5(f_n)$.

Remark 5.9 *All the covers f_n in Proposition 5.8 have* even *degree.*
Can one find covers of odd degree with equal branch points?

5.5 Towers of unramified extensions

Question: Can we compose covers from C to E to get towers of regular unramified Galois covers over finite extensions of K?

For this it is necessary (and also sufficient (use our machinery of Hurwitz spaces over schemes) that we find infinitely many minimal covers with the same *branch points*.

Because of Proposition 5.7 we have to restrict ourselves to fields of positive characteristic $p > 0$.

For *finite fields* K we can use arithmetic properties of modular forms (done by **Kiming**) to give an affirmative answer.

If K is not a finite field we can use Proposition 5.8.
We get

Proposition 5.10 ***(Diem-F)*** *Let K be a field of odd positive characteristic. Let E be an elliptic curve over K.*
Then there is a curve C of genus 2 covering E which has an infinite unramified regular Galois pro-cover defined over a finite extension of K.