Curves of Genus 2 with Elliptic Differentials and Related Hurwitz Spaces

Gerhard Frey
IEM
University of Duisburg-Essen

SCNTD
October 14, 2006
This is joint work with

E. Kani

and, partly, with

C. Diem

and

H. Völklein
# Contents

1  **Isomorphic Torsion Structures**  
   6

2  **Hurwitz Spaces**  
   9
   2.1  A “special” Hurwitz surface  
   9
   2.2  Covers of Elliptic Curves by Curves of Genus 2  
   13

3  **Moduli Functors**  
   20
   3.1  The “Basic Construction”  
   20
   3.2  Twisted Modular Curves  
   22
   3.3  Varying the Base Curve  
   25
   3.4  Diagonal Surfaces  
   28
   3.5  Back to the Hurwitz World  
   30

4  **Compactification**  
   32
   4.1  The boundary curves of $H'_n$  
   35
   4.2  The Rigidity Number  
   38

5  **Rational Points**  
   40
   5.1  The Conjectures Restated  
   40
   5.2  Points related to Isogenies  
   43
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.2.1</td>
<td>Universal construction</td>
<td>46</td>
</tr>
<tr>
<td>5.3</td>
<td>Degree of covers</td>
<td>47</td>
</tr>
<tr>
<td>5.4</td>
<td>Ramification points</td>
<td>49</td>
</tr>
<tr>
<td>5.5</td>
<td>Towers of unramified extensions</td>
<td>53</td>
</tr>
</tbody>
</table>
$K$ is a field of characteristic $p \geq 0$, $p \neq 2$
of finite type over its prime field $K_0$
with absolute Galois group $G_K$.

$n \in \mathbb{N}$ is always assumed to be prime to $p$.
The representation induced by the action of $G_K$ on the $n$-torsion points $E[n]$ of an elliptic curve $E$ over $K$ is denoted by $\rho_{E,n}$
1 Isomorphic Torsion Structures

Conjecture 1.1 Darmon:
There is a number $n_0 = n_0(K)$ such that for all $E, E'$ over $K$ and all $n \geq n_0$ we have
\[ \rho_{E,n} \cong \rho_{E',n} \]
iff
\[ E \text{ is } K\text{-isogenous to } E'. \]

Kani:
Maybe there are finitely many exceptions, and: for $K = \mathbb{Q}$ the bound $n_0 = 23$ suffices.
Much weaker is

**Conjecture 1.2** For all elliptic curves $E_0$ over $K$ there is a number $n_0 = n_0(K, E_0)$ such that for all $E$ defined over $K$ we get:

For $n \geq n_0$ it is equivalent

1. $\rho_{E,n} \cong \rho_{E_0,n}$
2. $E$ is isogenous to $E_0$.

For global fields (or more generally, fields with divisor theory satisfying finiteness conditions) this conjecture would follow from the **height conjecture** for elliptic curves.

Since the height conjecture is true over function fields the hard case is that $K$ is a number field.
Assume that $K = \mathbb{Q}$.

- The height conjecture is equivalent with the ABC-conjecture and with the degree conjecture for modular parameterizations: There exist $c, d \in \mathbb{R}_{>0}$ such that for minimal

$$\varphi : X_0(N_E) \rightarrow E$$

$$\log(\deg(\varphi)) \leq c \log(N_E) + d$$

- equivalences between Galois representations on torsion points of elliptic curves correspond to congruences of cusp forms.

- The Asymptotic Fermat Conjecture is equivalent with Conjecture 1.2 for even $n$.  

8
2 Hurwitz Spaces

2.1 A “special” Hurwitz surface

Look at covers
\[ \varphi : \mathbb{P}^1_K \to \mathbb{P}^1_K \]
of (for simplicity) odd degree \( n \) which are primitive (i.e. has no proper intermediate subcovers) and which have the following ramification behavior:

(*) \( \varphi \) is ramified in 5 points \( P_1, \ldots, P_5 \) with ramification order at most 2, and the ramification cycle corresponding to \( P_5 \) in the Galois closure of the cover is a transposition.

This condition clearly implies that \( P_5 \) has exactly one ramified extension \( Q_5 \) in the cover and that \( P_5, Q_5 \in \mathbb{P}^1(K) \).
Let \( r_i \) denote the number of unramified extensions of \( P_i \), for \( 1 \leq i \leq 4 \). Since

\[
-2 = -2n + 1 + \sum_{1 \leq i \leq 4} \frac{(n - r_i)}{2} = 1 - \sum_{1 \leq i \leq 4} \frac{r_i}{2}.
\]

we get \( \sum r_i = 6 \).

Since \( n \) is odd, \( r_i \) is odd and so there are

- \( P_1, P_2, P_3 \) with \( r_i = 1 \)
- \( P_4 \) with \( r_4 = 3 \)

It follows that \( P_4, P_5, P_1 + P_2 + P_3 \) and the discriminant divisor

\[
\text{disc}(\varphi) = P_1 + \ldots + P_5
\]

are \( K \)-rational.
The Galois closure $\overline{\varphi}$ of a cover $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ satisfying (\mbox{\ast}) has Galois group $S_n$ and has a fixed ramification cycle type $C$. So we get a moduli functor.

Call two covers

$$\varphi_1, \varphi_2 : \mathbb{P}^1 \to \mathbb{P}^1$$

to be weakly equivalent if there exist $\alpha_i \in \text{Aut}(\mathbb{P}^1)$ such that

$$\varphi_1 \alpha_1 = \alpha_2 \varphi_2.$$

The resulting quotient moduli functor is coarsely represented by the quotient space $H^*_n = H^{\text{in}}(S_n, C)/\text{Aut}(\mathbb{P}^1)$. (Fried, Völklein et al).

It is a (non-complete) irreducible surface.
Remark 2.1 (a) Since $H_n^*$ is only a coarse moduli space for the quotient functor, it is more difficult to characterize the $K$-rational points of $H_n^*$.

(b) The basic existence result (Riemann) yields that there always exist such covers (over algebraically closed ground fields of characteristic 0) for arbitrary points $P_1, \ldots, P_5$.

But it seems more difficult to determine explicitly how many such covers exist and to decide rationality questions over arbitrary ground fields. (Indeed, the underlying ramification cycle structure is highly non-rigid!)

Moreover, the constructions shed little light on the actual structure of the Hurwitz spaces.
2.2 Covers of Elliptic Curves by Curves of Genus 2

Why is $H_n^*$ interesting? It is characterized by

- covers of $\mathbb{P}^1$ of odd degree by itself
- with 5 ramification points
- one ramification cycle is a transposition.

**Trivial observation**

$P_1, \cdots, P_4$ determine an elliptic curve $E/K$ unique up to a quadratic twist. For $x$ of $\mathbb{P}^1_K$ with $(x)_{\infty} = P_4$ take

$$E : y^2 = f_3(x) \text{ with } (f_3)_0 = P_1 + P_2 + P_3.$$
Let $C = E \times_{\mathbb{P}^1} \mathbb{P}^1$ be the normalization of the fibre product of $E$ and $\mathbb{P}^1$ over $\mathbb{P}^1$ (with respect to the morphisms $\pi : E \to \mathbb{P}^1$ and $\varphi$).

$C$ is an irreducible curve of genus 2 satisfying a hyperelliptic equation of the form

$$Y^2 = f_3(X) \cdot g_3(X)$$

with $g_3$ a polynomial of degree 3 corresponding to the 3 unramified extensions of $P_4$ in the cover $\varphi$. 


$C$ comes together with morphisms

\[ f : C \to E \]

and

\[ \pi' : C \to \mathbb{P}^1. \]
Properties of $f$

1. The morphism $f : C \to E$ is minimal.

2. For the Weierstraß divisor $W_C = W_1 + \ldots + W_6$ we have

$$f^*W_C = 3 \cdot 0_E + P_1' + P_2' + P_3'.$$  \(1\)

3.

$$f \circ \omega_C = [-1]_E \circ f,$$  \(2\)

where $\omega_C$ denotes the hyperelliptic involution of $C$.

**Definition 2.2** A cover $f : C \to E$ with properties 1, 2 and 3 is called normalized.
By the Riemann-Hurwitz genus formula we see that the different of $f$ has to be a divisor of degree 2.

In our situation it is easily identified: The point $P_5$ has two distinct extensions $P$ and $P' = [-1]_E P$ to $E$, and there is exactly one point $Q$ resp. $Q' = \omega Q$ over $P$ resp. $P'$ which is ramified of order 2. Hence the discriminant divisor of $f$ is equal to $\pi^*(P_5)$.

Conversely, assume that

$$f_0 : C \to E$$

is a minimal cover of an elliptic curve $E$ by a curve of genus 2 of odd degree $n$ defined over $K$. 
Lemma 2.3 There is a unique translation $\tau : E \to E$ such that

$$f = \tau \circ f_0$$

is normalized.

$f$ factors over the hyperelliptic cover $\pi' : C \to C/\langle \omega \rangle = \mathbb{P}^1$. and induces a primitive cover $\varphi : C/\langle \omega \rangle = \mathbb{P}^1 \to E/\langle -\text{id}_E \rangle = \mathbb{P}^1$ of degree $n$ such that $\varphi \circ \pi' = \pi \circ f$.

Let $P_5 \in \mathbb{P}^1(K)$ be such that $\text{Disc}(f) = \pi^*(P_5)$, $P_1 + P_2 + P_3 + P_4 = \pi_*(E[2])$. Then

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^1$$

defines a point in $H^*_n$ iff $\text{Disc}(f)$ is reduced, i.e. if and only $P_5 \notin \{P_1, \ldots, P_4\}$ (generic case).
Consider the moduli problem $\mathcal{H}_n$ which classifies isomorphism classes of pairs $(C, f)$ where $C$ is a curve of genus 2 and $f$ is a normalized covering map from $C$ to an elliptic curve $E$ of degree $n$. Let $\mathcal{H}'_n$ denote the subproblem which classifies the covers for which $\text{Disc}(f)$ is reduced. We shall find surfaces $H_n$ and $H'_n$ which represent these problems coarsely.
3 Moduli Functors

3.1 The “Basic Construction”

**Observation:** A normalized cover
\[ f : C \to E \]

induces a map
\[ f_* : J_C \to E \]

whose kernel is an elliptic curve \( E' \subset J_C \) intersecting \( f^*(E) \) exactly in \( E'[n] \).

Hence it gives rise to a triple
\[ (E, E', \alpha_n : E[n] \to E'[n]) \]

over \( K \). \( J_C \) is (as p.p. variety)
\[ (E \times E')/\text{graph}(\alpha_n) \]

and \( \alpha_n \) is anti-isometric with respect to the Weil pairing.
Conversely: To a triple 

\((E, E', \alpha_n)\)

satisfying the conditions from above the abelian variety 

\((E \times E')/\text{graph}(\alpha_n)\)

has a principal polarization \(C\) (ie. a curve of genus 2) and a cover map 

\[ f : C \to E \]

of degree \(n\) which is normalized iff \(C\) is irreducible. It follows that the moduli problem \(\{(E, E', \alpha_n)\}\) has \(\mathcal{H}_n\) and \(\mathcal{H}'_n\) as subproblems.
3.2 Twisted Modular Curves

Now fix $E$ or, in the language of Hurwitz spaces, the points $P_1, \ldots, P_4$. So $P_5$ is varying. As result we get a curve, and, again in the language of covers of $E$, the “parameter” is the discriminant divisor.

More precisely:
For any extension field $L$ of $K$, let $E_L = E \otimes L$ denote the elliptic curve $E$ lifted to $L$. We now consider the set

$$\text{Cov}_{E/K,n}(L) := \{ f : C \to E_L : f \text{ normalized, defined over } L, \deg(f) = n \}/\text{Aut}E_L$$
The assignment $L \mapsto \text{Cov}_{E/K,n}(L)$ can be extended in a natural way to a functor $\mathcal{H}_{E/K,n} : \text{Sch}_K \to \text{Sets}$.

**Theorem 3.1 (Kani)**

If $n \geq 3$, then the functor $\mathcal{H}_{E/K,N}$ is finely represented by a smooth, affine and geometrically connected curve $H_{E/K,n}/K$ with the property that $H_{E/K,N} \otimes K_s$ is an open subset of the modular curve $X(N)/K_s$.

The proof of this theorem uses the basic construction.
Remark 3.2 The fact that the curve $H := H_{E/K,n}$ finely represents the functor $\mathcal{H}_{E/K,n}$ means that there exists a universal normalized genus 2 cover

$$f_H : C_H \rightarrow E \times H$$

of degree $n$ with the property that every normalized genus 2 cover $f : C \rightarrow E \times S$ of degree $n$ (where $S$ is any $K$-scheme) is obtained uniquely from $f_H$ by base-change. In particular, the set Cov$_{E/K,n}(K)$ of covers can be identified with the set of fibres $f_x := (f_H)_x : C_x \rightarrow E_x = E$ of $f_H$, where $x \in H(K)$. 

24
3.3 Varying the Base Curve

We want to represent the moduli functors $\mathcal{H}_n$ and $\mathcal{H}_n'$.
We can do this already for “fibers” (fixed choice of $P_1, \ldots, P_4$).
So we have to glue together.
But we can expect only coarse moduli schemes.
So a discussion of $K$-rational points is not enough, we shall have to study the associated moduli functors and/or stacks in more detail.
The appropriate frame is given by the concept of

*moduli problems for elliptic curves* introduced by Katz-Mazur.
It uses the moduli stack $\text{Ell}/R$ of all elliptic curves over a ring $R$ where the objects are relative elliptic curves over $R$-schemes $S$.

Moduli problems $\mathcal{P}$ give rise to contravariant functors

$$\tilde{\mathcal{P}} : \text{Sch}/R \to \text{Sets}$$

which classify isomorphism classes of $\mathcal{P}$-structures.
Example 3.3 We take $R = \mathbb{Z}[1/2n]$ and $P$ the functor $\mathbb{Z}_{n,-1}$ which associates to $E/S$ the set of elliptic curves defined over $S$ with isomorphic $n$-torsion structure with isotropic graph.

We know already that this functor is finely representable for fixed $E$ and get a moduli space $Z_{E,n,-1}$ defined over $R$. Take $j$ transcendental over $R$ and for $E_j$ a curve with invariant $j$. Define $Z_{E_j,n,-1} =: Z_{n,-1}$.

Theorem 3.4 (F-Kani) The moduli problems $\mathcal{H}_n$ and $\mathcal{H}'_n$ are coarsely representable by open subschemes $H_n$ and $H'_n$ of $Z_{n,-1}$ which is normal and affine and of relative dimension 1 over $\text{Spec}(\mathbb{Z}[1/2n][j]) = M(\Gamma[1])$. 
3.4 Diagonal Surfaces

For simplicity take $R = \mathbb{Z}[\frac{1}{2n}, \zeta_n]$. Take the modular affine curve

$$X(n)'/\mathbb{Z}[\frac{1}{2n}, \zeta_n]$$

with the action of

$$G_n = \text{SL}_2(\mathbb{Z}/n\mathbb{Z})/\{\pm 1\}.$$ 

$G_n \times G_n$ acts on the product

$$Y_n = X(n)'/\mathbb{Z}[\frac{1}{2n}, \zeta_n] \times \mathbb{Z}[\frac{1}{2n}, \zeta_n] X(n)'/\mathbb{Z}[\frac{1}{2n}, \zeta_n].$$
**Definition 3.5** Define

\[ \sigma_{-1} := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \]

Denote by \( \Delta_{-1} \) the graph of the conjugation by \( \sigma_{-1} \).

The quotient

\[ Z_{n,-1} = Y_n / \Delta_{-1} \]

is the modular diagonal quotient surface of type \((n, -1)\).

**Proposition 3.6** The modular diagonal quotient surface \( Z_{n,-1} \) is the coarse moduli space of the moduli problem \( Z_{n,-1} \) and hence the coarse moduli spaces \( H_n \) and \( H'_n \) are open subsets of \( Z_{n,-1} \).
3.5 Back to the Hurwitz World

We analyze the space $H^*_n$ by the same techniques (stacks) as above replacing $Ell/R$ by
the stack of curves of genus 0 over $R,M_0$ where $R$ is a ring in which $n!$ is invertible.
Let $X/S \in ob((M_0)/R)$ be a relative genus 0 curve, and consider the set
$$\text{Cov}_{X/S,n}^* := \{ \varphi : Y \to X : \varphi \text{ finite cover of degree } n$$
$$\text{whose fibres } \varphi_t \text{ satisfy } (*) \}/ \simeq X.$$ 
Then the rule $\mathcal{H}^*_n(X/S) = \text{Cov}^*_{X/S,n}$
defines a moduli problem on $(M_0)/R$,
i.e. a functor $\mathcal{H}^*_n : (M_0)/R \to \text{Sets}$. 

30
Theorem 3.7 The moduli functor $\mathcal{H}_n^*$ has a coarse moduli space. The rule
\[(C \xrightarrow{f} E) \mapsto (C/\langle \omega_C \rangle \to E/\langle [-1] \rangle)\]
defines a functor
\[q = q_n : \mathcal{H}'_n \to \mathcal{H}_n^*,\]
and the induced map
\[M(q) : H'_n = M(\mathcal{H}'_n) \to H^*_n = M(\mathcal{H}_n^*)\]
on the coarse moduli schemes is surjective and radical. Thus, if $R = K$ is a field of characteristic 0, then $M(q)$ is an isomorphism and hence
\[H'_n \simeq H^*_n\]
is an irreducible, normal affine surface.
4 Compactification

As the constructions of the previous section show, the moduli spaces $H'_n$ and $H^*_n$ are not compact. It is thus of interest to construct natural compactifications of these spaces and to investigate whether or not the boundary components have a modular interpretation in terms of covers of curves.

Since $H'_n$ was constructed as an open subset of the affine surface $Z_{n,-1}$, the natural compactification $\overline{Z}_{n,-1}$ of $Z_{n,-1}$ also serves as a compactification of $H'_n$. 
A modular interpretation can be obtained by studying the degeneration of the canonical compactification of the universal cover $C \to E_H$ over $H = H_{E/K,n}$ for each fixed elliptic curve $E/K$. On the other hand, the theory of Wewers et al gives a recipe for an (abstract) compactification of $H^{in}(S_n, \mathbb{C})$ in terms of covers, and this also gives a compactification of $H^*_n = H^{in}(S_n, \mathbb{C})/\text{Aut}(\mathbb{P}^1)$. 
Thus, in both interpretations the boundary curves of the Hurwitz spaces correspond to interesting degenerations of covers.

This leads to a nice picture about the interplay of geometry with group theory. We get a complete dictionary between degenerations with respect to the modular interpretation by covers of curves of genus 2 to elliptic curves (here level-$n$-structures and geometry give a complete classification), and on the other side with respect of degenerations of ramification points and ramification cycles (with a complete classification given by group theory (joint work with H. Völklein)). We shall give only a few hints.
4.1 The boundary curves of $H'_n$

The boundary curves naturally split into three types induced by the inclusions

$$H'_n \subset H_n \subset \mathbb{Z}_{n,-1} \subset \overline{\mathbb{Z}}_{n,-1}.$$ 

**Type 1:** $\partial_1 := H_n \setminus H'_n$

By the modular description of $H_n$ mentioned in subsection 2.2, the points of $\partial_1$ classify genus 2 covers $f : C \to E$ whose discriminant is not reduced, i.e. $\text{Disc}(f) = 2P$. We get $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ in which $P_5 = P_i$, for $i = 1, \ldots, 4$. Hence we can understand this degeneration as a coalescence process. Various possible ramification types can occur (predicted by group theory) and so $\partial_1$ decomposes further into several components consisting of curves with growing genus.
Type 2: $\partial_2 := Z_{n,-1} \setminus H_n$

It is immediate from the “basic construction” that the points in $\partial_2$ correspond to singular (stable) genus 2 curves whose Jacobian is smooth, i.e. to curves $C$ that are the union of two elliptic curves meeting in a single point. Again this means that $P_5$ moves to one of the points $P_1, \ldots, P_4$. 
Type 3: $\partial_3 := \overline{\mathbb{Z}}_{n,-1} \setminus \mathbb{Z}_{n,-1}$

$$\partial_3 = C_{\infty,1} \cup C_{\infty,2}$$

is the union of two irreducible curves $C_{\infty,i} \simeq X_1(n)$. The two cuspidal curves $C_{\infty,i}$ correspond to two different (sub)types of degenerations of covers $f : C \to E$. In the first type the curve $C$ degenerates to a singular curve of genus 2 (whose normalization is an elliptic curve), and in the second subtype the curve $E$ degenerates to singular curve of arithmetic genus 1 (whose normalization is therefore $\mathbb{P}^1$).

In terms of Hurwitz spaces this means that points in $P_1, \ldots, P_4$ coalesce.
4.2 The Rigidity Number

We shall give a finer application using geometry to get information about group theory and compute the “number of non-rigidity” of $H_n^*$.

**Theorem 4.1 (Kani)** Let $n > 1$ be an odd integer, and suppose that $K$ is an algebraically closed field. If either $\text{char}(K) = 0$ or $\text{char}(K) > n$ then the number of covers of degree $n$ ramified in $P$ is

$$\frac{1}{12}(n - 1)\#SL_2(\mathbb{Z}/n\mathbb{Z}).$$
Remark 4.2 This number is a measure of non-rigidity of the ramification type which defines $H_n^*$. Although this number can be directly defined via group theory, it seems very difficult, if not impossible, to compute this number in this way. The basic trick of the proof here is to compute instead a related number in the case of “degenerate covers”.
5 Rational Points

5.1 The Conjectures Restated

We come back to the conjectures stated in the first section.
We are looking for triples

$$(E, E', \alpha_n : E[n] \rightarrow E'[n])$$

defined over $K$.
We restrict ourselves for simplicity to the case that the determinant of $\alpha_n$ is $= -1$.
Conjecture 1.2 predicts that certain curves on $\mathbb{Z}_{n,-1}$, namely twisted modular curves, have, for $n$ large enough, only obvious points.
Conjecture 1.1 predicts that, for $n$ large enough, points in $Z_{n,-1}(K)$ correspond to pairs of elliptic curves $(E, E')$ which are isogenous.

This is supported by the following result of Hermann, Kani and Schanz.

**Proposition 5.1** For $n \leq 10$ the surface $Z_{n,-1}$ is rational or a $K - 3$-surface or an elliptic surface. For $n \geq 11$ it is a surface of general type.
Recall *Lang’s conjecture* which predicts that for such surfaces the $K$-rational points are concentrated on curves of genus $\leq 1$. Hence it is interesting to find “obvious” curves on $\mathbb{Z}_{n,-1}$ and then to prove that there are no other curves of low genus on this surface. Unfortunately, we do not see theoretical methods to come nearer to the second part of this task. Since the surfaces to be studied are given in a most explicit way one could think of doing computational experiments, i.e., for $n = 23$, the smallest interesting example.
5.2 Points related to Isogenies

The conjectures stated in Subsection 5.1 motivate to assume in the following that

\[ \eta : E \to E' \]

is a cyclic \( K \)–isogeny of minimal degree denoted by \( l_0 \).

To avoid trivial cases we always assume that \( l_0 > 1 \) and \( \eta \) is separable.

Denote by \( \alpha_n \) the restriction of \( \eta \) to \( E[n] \).

Of course, there may be other \( G_K \)-isomorphisms between \( E[n] \) and \( E'[n] \). We call triples \( (E, E', z \cdot \alpha_n) \) “generic” because of
Lemma 5.2 Assume that the centralizer of $G_K$ in $\text{Aut}(E_n)$ is $\mathbb{Z} \cdot \text{id}_{E[n]}$ and that $n$ is prime to $l_0$. Then every $G_K$-isomorphism between $E[n]$ and $E'[n]$ is of the form $z \cdot \alpha_n$ with $z \in \mathbb{Z}$ prime to $l_0$. In particular, this is the case if $E$ has no complex multiplication and $n$ is large enough (depending on $E$ and $K$).

An easy observation is

Proposition 5.3 For all $n$ prime to $l_0$ and all $z \in \mathbb{Z}$ prime to $n$ the abelian variety $J_n := (E \times E')/\text{graph}(z \cdot \alpha_n)$ is isomorphic to $E \times E'$. 
In order to get points on $H_n$ via the “basic construction” we need two additional properties:

Firstly the graph of $z \cdot \alpha_n$ has to be isotropic with respect to the Weil pairing and secondly one has to verify that the resulting curve $C$ is irreducible.

**Proposition 5.4** Assume that $l_0$ is square free and that $n$ is prime. Assume in addition that $E$ has no complex multiplication. Then there is an element $z \in \mathbb{Z}$ such that $z \cdot \alpha_n$ induces a covering

\[ C \to E \]

of degree $n$ iff $n = 2$ or $n$ is split into two non principal prime ideals in $\mathbb{Q}(\sqrt{-l_0})$. 

45
5.2.1 Universal construction

Take $l_0$ squarefree such that $\mathbb{Q}(\sqrt{-l_0})$ has class number $> 1$.
Let $F_0 := K_0(j, j_0)$ be the function field of $X_0(l_0)/K_0$.
Let $E_j$ be an elliptic curve with invariant $j$ defined over $F_0$.
Then there is a curve $C$ of genus 2 defined over $F_0$ and infinitely many numbers $n$ such that $C$ is covering $E_j$ of degree $n$. If $l_0$ is odd we get curves birationally equivalent to $X_0(l_0)$ on the Hurwitz spaces $H'_n$ resp. $H^*_n$.
For instance, take $l_0 = 5$. Then $X_0(l_0)$ has genus 0 and so for infinitely many $n$ we get a rational curve in $H'_n$ and $H^*_n$. 
5.3 Degree of covers

We have seen in Subsection 5.2.1 that under appropriate conditions there are fields of finite type $K$ and pairs $(E, E')$ of isogenous elliptic curves which are $K$-rationally covered by curves $C$ of genus 2 with Jacobian variety $E \times E'$. In fact, there are infinitely many such covers $f_n$ (if there is one) with the same cover curve $C$.

It is an interesting task to determine the degrees of these covers. It leads to questions about representation of numbers by quadratic forms.

We do not go into details in general but restrict ourselves to a special case.
Let \( \hat{\eta} \) be dual to the isogeny \( \eta : E \to E' \) of degree \( l_0 \).

Let \( \pi : C \to E \) and \( \pi' : C \to E' \) be covers of degree \( n \) with \( J_C = E \times E' \)
and
\[
\pi_\ast \circ \pi'_\ast = 0_E \text{ and } \pi'_\ast \circ \pi_\ast = 0_{E'}
\] .

\textbf{Lemma 5.5 (Diem-F)}

For \( a, b \in \mathbb{Z} \)

\[
a\pi + b\hat{\eta}\pi' : C \longrightarrow E
\]

has

\[
\deg(a\pi + b\hat{\eta}\pi') = na^2 + nl_0b^2.
\]

If \( a = 1 \) and \( b \equiv 0 \mod n \) the cover is \textit{minimal}. 

48
5.4 Ramification points

We continue to assume that $E$ and $E'$ are isogenous curves with a $K$-rational isogeny $\eta$ of minimal odd degree $l_0$ and that the $j$-invariant of $E$ is not contained in the prime field $K_0$ of $K$ and that $f : C \to E$ is induced by the universal construction described in Subsection 5.2.1. It follows that there are two ramification points of $f$ on $C$ and two branch points of $f$ on $E$ corresponding to a point $P_5(f) \in \mathbb{P}^1(K)$ with unique ramified extension $Q_5(f) \in \mathbb{P}^1(K)$.

We have seen that there are infinitely many covers $f_n : C \to E$ of different degrees $n$. 
Question:
How do $P_5$ resp. $Q_5$ move when we change $f$ (with fixed $C$).

Proposition 5.6 $Q_5(f)$ corresponds to an ideal in $K_0[X]$ of degree bounded by a number depending only on $E$.

The proof is not trivial; it uses results from arithmetical geometry of fibred surfaces.

This result could indicate finiteness properties of ramification resp. branch points of covers.

We remark that finiteness of $\{P_5(f_n)\}$ yields finiteness of $\{Q_5(f_n)\}$ since $K$—rationality of $P_5$ implies $K$-rationality of the corresponding ramification point on $C$. 
A closer look (F-Diem) reveals that the worlds in characteristic 0 and characteristic $> 0$ behave differently.

**Proposition 5.7** Let $K$ be a field of characteristic 0. Let $c_1$ and $c_2$ be minimal covers from $C$ to $E$ with $c_1 \neq \pm c_2$. Then the ramification loci of $c_1$ and $c_2$ are disjoint. So, if $\{f_n; n \in I\}$ is an infinite set of minimal covers from $C$ to $E$ then the set of ramification loci and the set of branch loci of these covers are infinite.
Proposition 5.8 Assume that $\text{char}(K) = p > 0$. Then all ramification points for all minimal covers from $C$ to $E$ lie in a finite set of order $\leq p^{\dim_Z \text{Hom}_K(J_C,E)}$, and there is a cover

$$C \to E$$

of degree 2 such that there are infinitely many covers

$$f_n : C \to E$$

defined over $K$ with $P_5(f) = P_5(f_n)$.

Remark 5.9 All the covers $f_n$ in Proposition 5.8 have even degree. Can one find covers of odd degree with equal branch points?
5.5 **Towers of unramified extensions**

**Question:** Can we compose covers from $C$ to $E$ to get towers of regular unramified Galois covers over finite extensions of $K$?

For this it is necessary (and also sufficient (use our machinery of Hurwitz spaces over schemes) that we find infinitely many minimal covers with the same branch points.

Because of Proposition 5.7 we have to restrict ourselves to fields of positive characteristic $p > 0$. 
For finite fields $K$ we can use arithmetic properties of modular forms (done by Kiming) to give an affirmative answer.

If $K$ is not a finite field we can use Proposition 5.8. We get

**Proposition 5.10 (Diem-F)** Let $K$ be a field of odd positive characteristic. Let $E$ be an elliptic curve over $K$.

Then there is a curve $C$ of genus 2 covering $E$ which has an infinite unramified regular Galois pro-cover defined over a finite extension of $K$. 