Special Riemann-Hurwitz Formulas in Iwasawa Theory

Jordan C. Schettler jschettler@math.arizona.edu

THE UNIVERSITY OF ARIZONA

Analogies

Function Fields k over $\mathbb{F}_q$	Number Fields k
$[k : \mathbb{F}_q(t)] < \infty$, $\overline{\mathbb{F}}_q \cap k = \mathbb{F}_q$	$[k : \mathbb{Q}] < \infty$
k is the function field of a projective curve X_k over $\mathbb{F}_q$ (regular, integral)	k is the function field of $X_k := \mathrm{Spec}(\mathcal{O}_k)$, $\dim = 1$ (regular, integral)
$\exists\zeta_k(s)$, Riemann Hyp.	$\exists\zeta_k(s)$, Riemann Hyp.

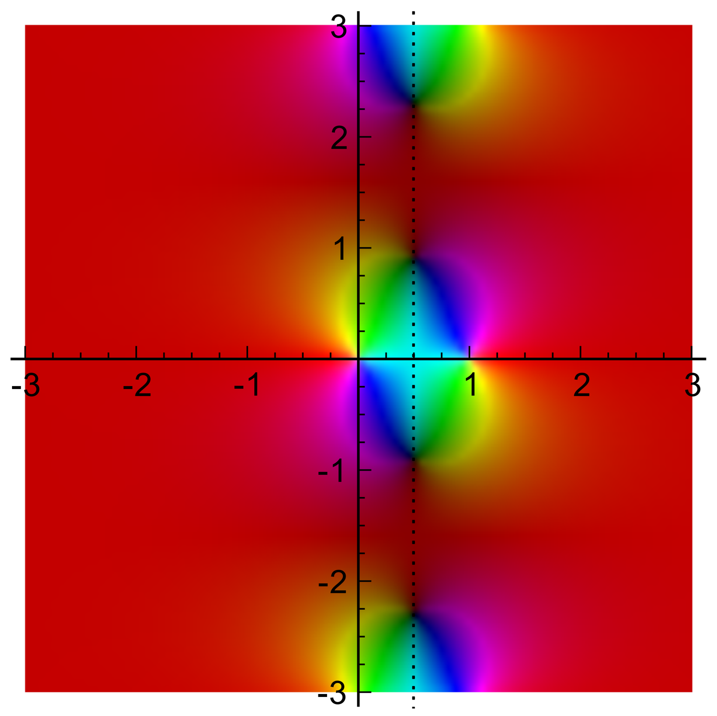
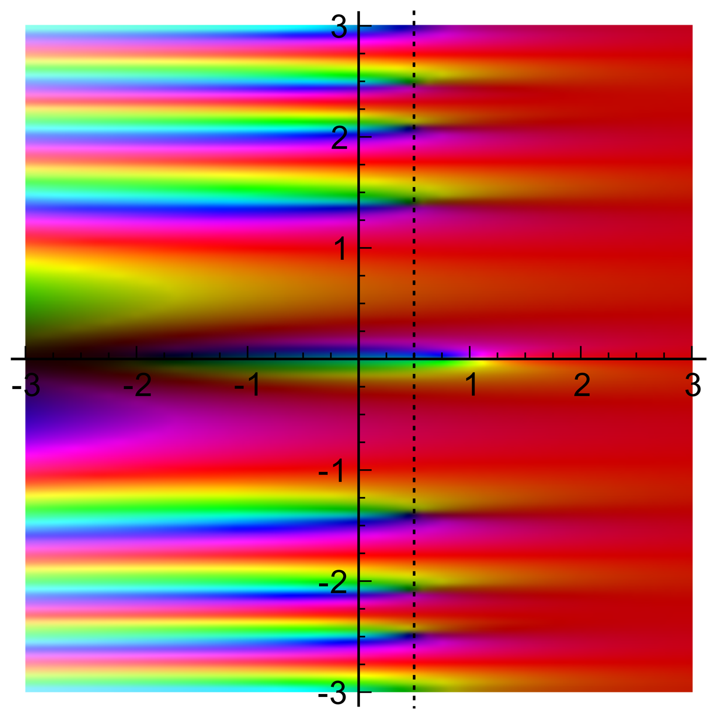
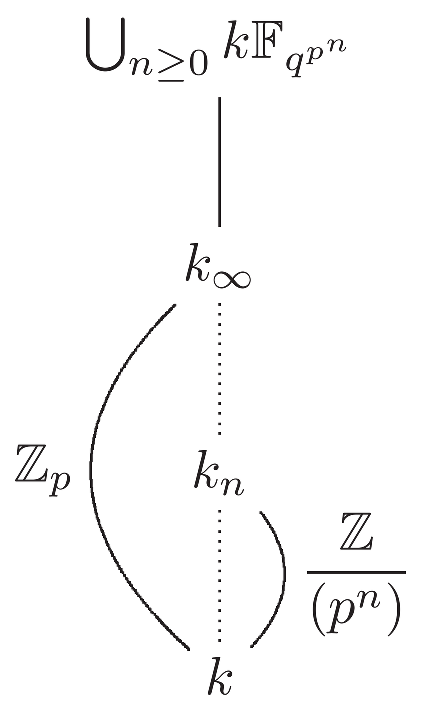
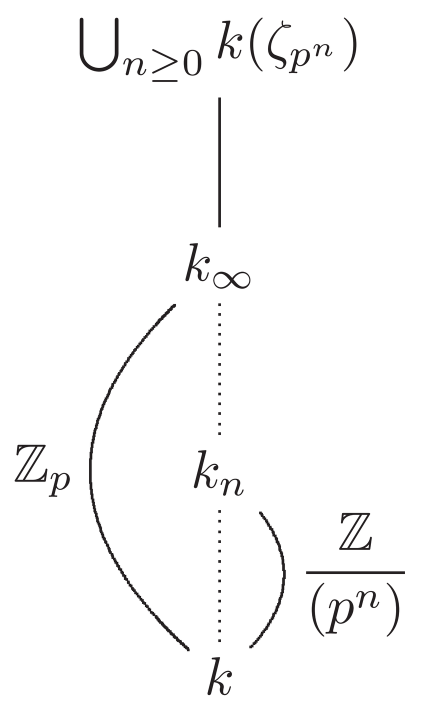

	

Figure: $Z(7^{-s})$ for a curve$/\mathbb{F}_7$	Figure: Riemann zeta $\zeta(s)$
$h(k) = \lvert\mathrm{Pic}^0(X_k)\rvert < \infty$	$h(k) = \lvert\mathrm{Pic}(X_k)\rvert < \infty$
	
Growth Formula: $\exists\lambda, \nu \in \mathbb{Z}$ s.t. $\forall n \gg 0$	Growth Formula: $\exists\lambda, \mu, \nu \in \mathbb{Z}$ s.t. $\forall n \gg 0$
$\mathrm{ord}_p h(k_n) = \lambda n + \nu$	$\mathrm{ord}_p h(k_n) = \lambda n + \mu p^n + \nu$

Remark:
Iwasawa conjectured that $\mu = 0$ for the $\mathbb{Z}_p$-extensions k_∞/k as above. Here we assume $\mu = 0$ for such extensions. A field $K = k_\infty$ for some number field k is called a $\mathbb{Z}_p$-**field**; equivalently, $[K : \mathbb{Q}_\infty] < \infty$. We take λ_K to be the λ in the growth formula for K/k since it does not depend on k.

Known Formulas

Let $[K : \overline{\mathbb{F}}_q(t)] < \infty$ with $p \nmid q$. Then K is the function field of a projective curve X_K (regular, integral) over $\overline{\mathbb{F}}_q$ and

$$\mathrm{Pic}^0(X_K)[p^\infty] \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{2g_K}$$

(Hurwitz): $\mathrm{Gal}(L/K) \cong \mathbb{Z}/(p) \Rightarrow$

$$2g_L = p2g_K - (p-1)2 + \sum_{x\in X_L} (e_x - 1)$$

Let $[K : \mathbb{Q}_\infty] < \infty$ with $\mu = 0$. Then K is the function field of $X_K = \mathrm{Spec}(\mathcal{O}_K[1/p])$ (regular, integral) with $\dim = 1$ and

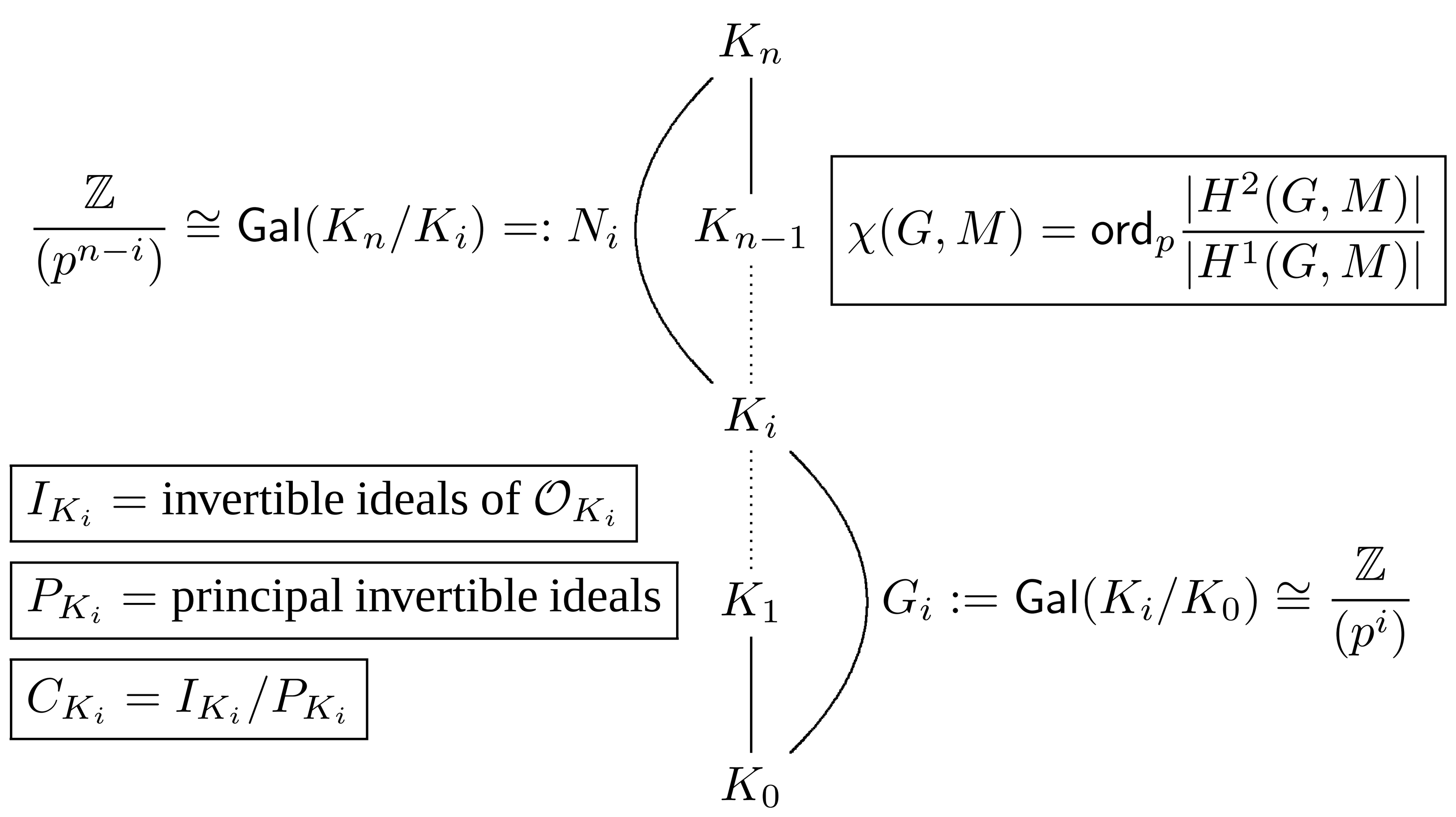
$$\mathrm{Pic}(X_K)[p^\infty] \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{\lambda_K}$$

(Iwasawa): $\mathrm{Gal}(L/K) \cong \mathbb{Z}/(p) \Rightarrow$

$$\lambda_L = p\lambda_K - (p-1)\chi(G, P_L) + \sum_{x\in X_L} (e_x - 1)$$

Special Formulas for $\mathbb{Z}_p$-Fields

Notation:
Let $[K_0 : \mathbb{Q}_\infty] < \infty$ with $\mu = 0$. Consider a tower:



Main Result:

$$\frac{\lambda_{K_n} - p^n\lambda_{K_0}}{p-1} = p^{n-1}\chi(G_n, C_{K_n}) - \sum_{i=1}^{n-1} \varphi(p^i)\chi(G_i, C_{K_i})$$

$$= \frac{p^n}{np - n + 1}\chi(N_0, C_{K_n}) + \sum_{i=1}^{n-1} \frac{p^i(p-1)\chi(N_{n-i}, C_{K_n})}{(ip - i + p)(ip - i + 1)}$$

Applications of the Special Formulas

Extending Ferrero's and Kida's Computations:
Let K be the cyclotomic $\mathbb{Z}_2$-extension of the first layer k in the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ where p is 2 or a Fermat prime and $h(k)$ is odd (e.g., $p = 2, 3, 5, 17, 257, \ldots$), and let L be the cyclotomic $\mathbb{Z}_2$-extension of $k(\sqrt{-d})$ with $d \in \mathbb{Z}$ squarefree and $d > 2 \geq (d, p)$. Then

$$\lambda_L = |S| - 1$$

where S is the set of finite places of L not lying above 2 which are ramified in L/K.

A Vanishing Criterion:
Let L/K be a cyclic p-extension of $\mathbb{Z}_p$-fields which is unramified at every infinite place. Suppose $K = k_\infty$ for a number field k having exactly one place above p with $p \nmid h(k)$. Then $\lambda_L = 0$ if and only if p does not divide the orders in C_L of the classes of finite places not lying above p and ramified in L/K.

Congruences:

$$\lambda_{K_n} \equiv \lambda_{K_i} \pmod{\varphi(p^{i+1})} \quad \text{for all } i = 0, \ldots, n$$

$$\lambda_{K_n} \equiv -p^{n-1}\chi(G_n, C_{K_n}) - (p-1)\sum_{i=1}^{n-1} \varphi(p^i)\chi(G_i, C_{K_i}) \pmod{p^n}$$

and if $p \nmid n - 1$

$$\lambda_{K_n} \equiv \sum_{i=1}^{n-1} \frac{p^i(p-1)^2\chi(N_{n-i}, C_{K_n})}{(ip - i + p)(ip - i + 1)} \pmod{p^n}$$

Note: The first congruence above can be thought of as an analog to the corresponding congruences for twice the genera of function fields in cyclic p-extensions. For example, the genus of the Fermat curve $x^d + y^d = z^d$ over $\mathbb{C}$ is $(d-1)(d-2)/2$, so we can see a special case of the congruence quite easily by noticing that

$$2\frac{(p^n - 1)(p^n - 2)}{2} \equiv 2\frac{(p^i - 1)(p^i - 2)}{2} \pmod{\varphi(p^{i+1})}.$$

Inequalities:

$$\mathrm{ord}_p\lvert H^2(G_n, P_{K_n})\rvert \leq n\lambda_{K_0} + \mathrm{ord}_p\lvert H^1(G_n, P_{K_n})\rvert + \chi(G_n, I_{K_n})$$

and if $\lambda_{K_0} = 0$

$$\mathrm{ord}_p\lvert H^2(G_n, P_{K_n})\rvert \leq \chi(G_n, I_{K_n})$$