“Distribution of modular symbols and L-values: computations and applications”, a workshop at Harvard, May 11-12, 2017

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For some more details and data concerning the discussion below, see the slides at:

- `http://www.math.uci.edu/~krubin/lectures/BIRS_2016.pdf`
- `http://www.math.harvard.edu/~mazur/papers/heuristics.Toronto.12.pdf`

The goal of the workshop is to try to begin to understand the growth of Mordell-Weil groups of elliptic curves in infinite abelian extensions, using heuristics and calculations. For example, consider the following conjecture.

Conjecture 1. *Let $K/\mathbf{Q}$ be an abelian extension of $\mathbf{Q}$ containing only finitely many extensions of $\mathbf{Q}$ of degrees $2, 3, 5$. If E is an elliptic curve over $\mathbf{Q}$, then $E(K)$ is finitely generated.*

Conjecture 1 is a slight sharpening of a conjecture of David, Fearnley and Kisilevsky [**DFK**]. They used random matrix heuristics (and some supporting computations) to estimate the probability that a twisted L-value $L(E, \chi, 1)$ vanishes.

Our proposed approach is meant to complement theirs by going directly to the expression for $L(E, \chi, 1)$ in terms of modular symbols, using experimentally observed properties of the distribution of modular symbols to estimate the probability that $L(E, \chi, 1) = 0$.

Conjecture 2. *Fix an elliptic curve $E/\mathbf{Q}$, and a divisor κ of the conductor N_E. There are constants $\alpha(E)$ and $\beta(E, \kappa)$ such that as m goes to infinity among positive integers with $\gcd(N_E, m) = \kappa$, the distribution of values of modular symbols*

$$\left\{ \left[\tfrac{a}{m}\right]^{\pm}_{E} : 1 \le a \le m, \gcd(a, m) = 1 \right\}$$

is asymptotic to a normal distribution with variance $\alpha(E)\log(m) + \beta(E, \kappa)$.

We have an explicit formula for $\alpha(E)$ (it is essentially $L(\mathrm{Sym}^2(E), 1)$), but $\beta(E, \kappa)$ is more mysterious.

Remark 3. A slightly weaker version of Conjecture 2 is a theorem of Petridis & Risager [**PR**].

On the other hand, there are some regularities in the distribution of modular symbols. For example, the following conjecture fits extremely well with data (and has a fairly simple heuristic justification).

Conjecture 4 (Mazur-Rubin-Stein)**.** *For every real number $x \in [0,1]$ and $m \in \mathbf{Z}_{>0}$ define*

$$G^+_{E,m}(x) := \frac{1}{m} \sum_{a=0}^{\lfloor mx \rfloor} \left[\tfrac{a}{m}\right]^+_E, \quad g_E(x) := \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{a_n}{n^2} \sin(\pi n x).$$

Then $G^+_{E,m}$ converges uniformly to $g_E(x)$ as m goes to infinity.

Definition 5. Suppose $E/\mathbf{Q}$ is an elliptic curve and $L/\mathbf{Q}$ is a cyclic extension. Define the *θ-element*

$$\theta_{E;L/\mathbf{Q}} := \sum_{g \in G} c_{E;L/\mathbf{Q}}(g)[g] \in \mathbf{R}[G]$$

characterized by the property that for all characters $\chi : G \to \mathbf{C}^\times$,

$$\chi(\theta_{E;L/\mathbf{Q}}) = \frac{\tau(\chi) L(E, \bar{\chi}, 1)}{\Omega_E}$$

where $\tau(\chi)$ is a Gauss sum and Ω_E is the appropriate period of E.

We call the $c_{E;L/\mathbf{Q}}(g)$'s the *θ-coefficients*, they are easily expressed as sums of modular symbols with denominator $\mathrm{cond}(L/\mathbf{Q})$. If the $c_{E;L/\mathbf{Q}}(g)$ behaved like *random* sums of modular symbols with denominator $\mathrm{cond}(L/\mathbf{Q})$, then we would understand their distribution and we could make a prediction for how often $L(E, \chi, 1) = 0$.

Consider the normalized θ-coefficients

$$c'_{E;L/\mathbf{Q}}(g) := \frac{c_{E;L/\mathbf{Q}}(g)}{\sqrt{(\alpha(E)\log(m) + \beta(E, \gcd(N_E, m)))\varphi(m)/d}}$$

where $m := \mathrm{cond}(L/\mathbf{Q})$, $d := [L : \mathbf{Q}]$, and φ is the Euler φ-function. The normalization is so that if the c_E were random sums of $\varphi(m)/d$ modular symbols, then the $c'_{E;L/\mathbf{Q}}(g)$ would satisfy a normal distribution with variance 1. Experimentally, for fixed degree d and varying m, the $c'_{E;L/\mathbf{Q}}(g)$ *do* seem to converge to a distribution $\Lambda_{E,d}(t)$. However, at least for small d, this appears quite different from a normal distribution. For example, when $d = 3$ it is not even clear whether or not $\Lambda_{E,d}(t)$ is bounded near $t = 0$.

If the limiting distribution $\Lambda_{E,d}(t)$ exists and is bounded, then no matter what its shape is, this leads to a prediction that the number $N(d, X)$ of characters χ of order d with $\mathrm{cond}(\chi) < X$ and $L(E, \chi, 1) = 0$ satisfies

$$N(d, X) \ll \frac{X^{1-\varphi(d)/4}}{\log(X)^{\varphi(d)/4}}. \tag{1}$$

The bound in (1) (and even something substantially weaker) is the motivation for Conjecture 1. However, the bound (1) is not true when $d = 3$, because there are families [**FKK**] showing that

$$N(3, X) \gg \sqrt{X}.$$

But if $\Lambda_{E,3}(t)$ grows like a suitable power of $|\log(t)|$ near $t = 0$, then our upper bound would give the answer predicted by [**DFK**].

The discussion above leads to the following projects for discussion during the workshop.

- Extend the computations of θ coefficients, to try to understand the limiting distributions $\Lambda_{E,d}(t)$.

- Why are the $\Lambda_{E,d}(t)$ not normal distributions, i.e., why are the θ-coefficients not random sums of modular symbols?

- Are there more constructions of rational points that give lower bounds for $N(d, X)$, like the one in [**FKK**] for $d = 3$, and one due to Darmon when $d = 5$ and $\mathbf{Q}$ is replaced by a quadratic extension? (Darmon's construction shows that a simple generalization of the [**DFK**] conjecture from $\mathbf{Q}$ to quadratic fields cannot hold for $d = 5$.)

- How much of this discussion carries over to the case where $\mathbf{Q}$ is replaced by an arbitrary base number field?

References

[**DFK**] C. David, J. Fearnley, H. Kisilevsky, Vanishing of L-functions of elliptic curves over number fields. In: Ranks of elliptic curves and random matrix theory, *London Math. Soc. Lecture Notes* **341**, Cambridge Univ. Press, Cambridge (2007) 247–259.

[**FKK**] J. Fearnley, H. Kisilevsky, M. Kuwata, Vanishing and non-vanishing Dirichlet twists of L-functions of elliptic curves. *J. London Math. Soc.* **86** (2012) 539-557.

[**PR**] Y. Petridis, M. Risager, Modular symbols have a normal distribution. *Geometric and Functional Analysis* **14** (2004) 1013–1043.