GENERALISED ELLIPTIC BOUNDARY PROBLEMS

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Abstract. For elliptic systems of differential equations on a manifold with boundary, we prove the Fredholm property of a class of boundary problems which do not satisfy the Shapiro-Lopatinskii property. We name these boundary problems generalised elliptic, for they preserve the main properties of elliptic boundary problems. Moreover, they reduce to systems of pseudodifferential operators on the boundary which are generalised elliptic in the sense of Saks (1997).

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Introduction

Nowadays by ellipticity is usually meant the property of operators in algebras with symbolic structure to have an invertible symbol. While a symbol may characterize a very particular property of operators, one tries to construct a full set of symbols which control the Fredholm property of operators. For algebras of pseudodifferential operators on spaces with singularities the invertibility of symbols no longer can be verified effectively at all.

The simplest singularity might occur is the boundary of a compact C^∞ manifold. The Fredholm property of boundary value problems in Sobolev spaces on such manifolds is equivalent to the invertibility of two symbols. One of the two symbol maps is given by ordinary differential operator which acts in spaces of bounded functions on the semiaxis under suitable conditions at the origin. Its invertibility is referred to as the Shapiro-Lopatinskii condition.

The symbol construction is related to an appropriate choice of the principal parts of operators. These are in turn determined by available group actions on the underlying manifold. The operators are given domains prescribed by the group actions in question. The construction of abstract Sobolev spaces based on a group action goes as far as [Sch91].

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Generally, the ellipticity property of a pseudodifferential operator is only a property of the way in which the operator is written, and it may appear or disappear by replacing the operator by a homotopically equivalent operator. On the other hand, the Fredholm property of a pseudodifferential operator is homotopically invariant but it may be necessary to change the function spaces for the evaluation of kernel and cokernel.

If motivated by the Fredholm property, the class of elliptic operators should therefore survive under homotopical equivalence. In this way we obtain what we call generalised elliptic operators. More precisely, a pseudodifferential operator is called generalised elliptic if it is homotopically equivalent to a classical elliptic pseudodifferential operator.

We show that generalised elliptic operators give rise to Fredholm operators in suitable function spaces.

1. Homotopical equivalence

In this section we recall the notion of equivalent overdetermined pseudodifferential operators which belong to a calculus, cf. § 1.2.1 in [Tar95]. By an overdetermined pseudodifferential operator is meant a sequence

$$C^\infty(E^0) \stackrel{d_E^0}{\to} C^\infty(E^1) \stackrel{d_E^1}{\to} C^\infty(E^2) \tag{1.1}$$

consisting of two pseudodifferential operators d_E^0 and d_E^1 satisfying $d_E^1 d_E^0 = 0$. We write it d_E for short.

Two overdetermined pseudodifferential operators d_E and d_F are called equivalent if there exist pseudodifferential operators M_i of type $F^i \to E^i$ and M_i^{-1} of type $E^i \to F^i$, for $i = 0, 1, 2$, and pseudodifferential operators h_i^E of type $E^i \to E^{i-1}$ and h_i^F of type $F^i \to F^{i-1}$, for $i = 1, 2$, with the property that the following conditions are fulfilled:

$$\begin{array}{rrcl}
1) & M_{i+1} d_F^i - d_E^i M_i & = & 0, \\
 & M_{i+1}^{-1} d_E^i - d_F^i M_i^{-1} & = & 0;
\end{array}
\qquad
\begin{array}{rrcl}
2) & M_i^{-1} M_i & = & I - h_{i+1}^F d_F^i - d_F^{i-1} h_i^F, \\
 & M_i M_i^{-1} & = & I - h_{i+1}^E d_E^i - d_E^{i-1} h_i^E
\end{array}$$

for $i = 0, 1$, cf. the diagram

$$\begin{array}{ccccc}
C^\infty(F^0) & \underset{h_1^F}{\overset{d_F^0}{\rightleftarrows}} & C^\infty(F^1) & \underset{h_2^F}{\overset{d_F^1}{\rightleftarrows}} & C^\infty(F^2) \\
{\scriptstyle M_0}\big\downarrow\big\uparrow{\scriptstyle M_0^{-1}} & & {\scriptstyle M_1}\big\downarrow\big\uparrow{\scriptstyle M_1^{-1}} & & {\scriptstyle M_2}\big\downarrow\big\uparrow{\scriptstyle M_2^{-1}} \\
C^\infty(E^0) & \underset{h_1^E}{\overset{d_E^0}{\rightleftarrows}} & C^\infty(E^1) & \underset{h_2^E}{\overset{d_E^1}{\rightleftarrows}} & C^\infty(E^2).
\end{array} \tag{1.2}$$

If two overdetermined pseudodifferential operators d_E and d_F are equivalent, then there is a topological isomorphism between the cohomology spaces of complex (1.1) and that for d_F at steps 0 and 1. This isomorphism is defined by the operators M_0 and M_1 from (1.2).

Theorem 1.1. *Let d_E and d_F be equivalent overdetermined pseudodifferential operators. Then the cohomology spaces $H^i(C^\infty(E^\cdot))$ and $H^i(C^\infty(F^\cdot))$ are topologically isomorphic for $i = 0, 1$.*

In particular, if both d_E^1 and d_F^1 vanish, then $d_F^0 : C^\infty(F^0) \to C^\infty(F^1)$ is Fredholm if and only if so is the operator $d_E^0 : C^\infty(E^0) \to C^\infty(E^1)$.

Proof. Denote by $N(d_F^0)$ the null-space of d_F^0, and $N(d_E^0)$ the null-space of d_E^0. We claim that these spaces are topologically isomorphic. To prove this, consider a map of $N(d_F^0)$ to $N(d_E^0)$ given by $v \mapsto M_0 v$ for $v \in N(d_F^0)$. By 1),

$$\begin{aligned} d_E^0 M_0 v &= M_1 d_F^0 v \\ &= 0, \end{aligned}$$

hence $M_0 v$ belongs to $N(d_E^0)$ indeed. Suppose that $v \in N(d_F^0)$ satisfies $M_0 v = 0$. Then by 2) we get

$$\begin{aligned} v &= M_0^{-1} M_0 v + h_1^F d_F^0 v \\ &= 0, \end{aligned}$$

i.e., the map $v \mapsto M_0 v$ is injective. Finally, let $u \in N(d_E^0)$. Set $v = M_0^{-1} u$, then $v \in C^\infty(F^0)$ and

$$\begin{aligned} d_F^0 v &= M_1^{-1} d_E^0 u \\ &= 0, \end{aligned}$$

which is due to 1). Moreover, by 2), we get

$$\begin{aligned} M_0 v &= u - h_1^E d_E^0 u \\ &= u, \end{aligned}$$

i.e., the map $v \mapsto M_0 v$ is surjective.

We now denote by $R(d_F^0)$ the range of d_F^0, and $R(d_E^0)$ the range of d_E^0. We claim that the cohomology spaces $N(d_F^1)/R(d_F^0)$ and $N(d_E^1)/R(d_E^0)$ are topologically isomorphic. To prove this, consider a map of $N(d_F^1)/R(d_F^0)$ to $N(d_E^1)/R(d_E^0)$ given by $[g] \mapsto [M_1 g]$ for $g \in N(d_F^1)$, where by $[g]$ is meant the cohomology class containing g, and similarly for $[M_1 g]$. This map is well defined, for

$$\begin{aligned} d_E^1 M_1 g &= M_2 d_F^1 g \\ &= 0, \end{aligned}$$

the second equality being due to 1), and if $g = d_F^0 v$ for some $v \in C^\infty(F^0)$, then $M_1 g = d_E^0 M_0 v$ by 1), whence $[M_1 g] = 0$. Suppose $g \in N(d_F^1)$ satisfies $[M_1 g] = 0$, i.e., $M_1 g = d_E^0 u$ for some $u \in C^\infty(E^0)$. By 2), we obtain

$$\begin{aligned} g &= M_1^{-1} M_1 g + h_2^F d_F^1 g + d_F^0 h_1^F g \\ &= d_F^0 h_1^F g, \end{aligned}$$

showing $[g] = 0$, i.e., the map $[g] \mapsto [M_1 g]$ is injective. Finally, let $f \in N(d_E^1)$. Set $g = M_1^{-1} f$, then $g \in C^\infty(F^1)$ and

$$\begin{aligned} d_F^1 g &= M_2^{-1} d_E^1 f \\ &= 0, \end{aligned}$$

which is due to 1). Moreover, by 2), we get

$$\begin{aligned} M_1 g &= f - h_2^E d_E^1 f - d_E^0 h_1^E f \\ &= f - d_E^0 h_1^E f, \end{aligned}$$

i.e., $[M_1 g] = [f]$. Hence it follows that the map $[g] \mapsto [M_1 g]$ is surjective, which completes the proof. □

By this theorem we can see that equivalent overdetermined pseudodifferential operators are in fact different manifestations of a mathematical object describing the same physical or some other process.

The modern theory of operator algebras is mostly interested in describing the operators which are invertible modulo "small" operators. By "small" operators are usually meant compact operators, and then the class under study are Fredholm operators. On using representation theory one arrives at matrix or more general operator bundles where the invertibility is to be established. In this way one obtains what pretends to be an algebraic characterisation of the Fredholm property and is usually referred to as ellipticity. However, the ellipticity is strongly related to function spaces that are chosen to be the domain and target space of operators under consideration. In particular, the classical ellipticity corresponds to usual Sobolev spaces graded according to scalar orders of operators. This leads to numerous concepts of ellipticity.

Example 1.1. Consider the differential operator $Au := \operatorname{rot} u + u$ defined on functions $u : \mathbb{R}^3 \to \mathbb{R}^3$. It is not overdetermined, for the only differential operator A^1 satisfying $A^1 A = 0$ is obviously $A^1 = 0$. On completing the system $Au = 0$ to an involutive system we arrive at the overdetermined differential operator $B^0 = (1 \oplus \operatorname{div}) A$, i.e., $B^0 u = (Au, \operatorname{div} u)$, with a compatibility operator B^1 given by $B^1 f = \operatorname{div} f' - f_4$ for a function $f = (f', f_4)$ on $\mathbb{R}^3$ with values in $\mathbb{R}^4$. The differential operators A and B are homotopically equivalent, for the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & C^\infty(\mathbb{R}^3 \times \mathbb{R}^3) & \underset{0}{\overset{B^0}{\rightleftarrows}} & C^\infty(\mathbb{R}^3 \times \mathbb{R}^4) & \underset{h_2^F}{\overset{B^1}{\rightleftarrows}} & C^\infty(\mathbb{R}^3 \times \mathbb{R}) & \to & 0 \\
 & & {\scriptstyle 1}\big\downarrow\big\uparrow{\scriptstyle 1} & & {\scriptstyle M_1}\big\downarrow\big\uparrow{\scriptstyle M_1{}^{-1}} & & {\scriptstyle 0}\big\downarrow\big\uparrow{\scriptstyle 0} & & \\
0 & \to & C^\infty(\mathbb{R}^3 \times \mathbb{R}^3) & \underset{0}{\overset{A}{\rightleftarrows}} & C^\infty(\mathbb{R}^3 \times \mathbb{R}^3) & \underset{0}{\overset{0}{\rightleftarrows}} & 0 & &
\end{array}
\tag{1.3}
$$

takes place with $M_1 f = f'$ for $f = (f', f_4) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^4)$, and $M_1^{-1} f = (f, \operatorname{div} f)$ for $f \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, and $h_2^F y = (0, -y)$ for $y \in C^\infty(\mathbb{R}^3 \times \mathbb{R})$. It is easy to see that the first line of (1.3) is an elliptic complex in the classical sense. On the other hand, the operator A is nor elliptic in the classical sense neither Douglis-Nirenberg elliptic, cf. [KST04]. Hence, under homotopical equivalence the property of being elliptic can hardly be traced in explicit form.

This example shows that the only way to algebraically recognise the Fredholm property of a square system is to bring it to a normal form. The normal form suggests also appropriate function spaces in which A behaves properly. The diagram (1.3) shows for instance that if $\mathcal{M}$ is a compact closed Riemannian manifold of dimension 3 and A is the differential operator on one-forms on $\mathcal{M}$ defined by $Au = du + *u$, where $*$ is the Hodge star operator, then A induces a Fredholm operator from $H^s(T^*\mathcal{M})$ to the subspace of $H^{s-1}(\Lambda^2 T^*\mathcal{M})$ consisting of all f with $df \in H^{s-1}(\Lambda^3 T^*\mathcal{M})$.

2. Normal form

Let $\mathcal{M}$ be a C^∞ manifold with boundary $\partial\mathcal{M}$, and $\pi : V \to \mathcal{M}$ a vector bundle over $\mathcal{M}$.

Denote by $\pi^q : J_q(V) \to \mathcal{M}$ the bundle of q-jets of the bundle V. Introduce the canonical projections

$$\pi^q_r : J_q(V) \to J_r(V),$$

for $r \le q$, and define the embedding ε_q by requiring that the following complex be exact

$$0 \to S^q(T^*\mathcal{M}) \otimes V \xrightarrow{\varepsilon_q} J_q(V) \xrightarrow{\pi^q_{q-1}} J_{q-1}(V) \to 0.$$

Let s be a section of the bundle V. Then its qth prolongation, a section of $J_q(V)$, is denoted by $j^q s$. We write $\mathcal{S}(V)$ for a space of sections of the bundle V.

Definition 2.1. By a (partial) differential equation of order q on V is meant a subbundle $\mathcal{R}_q$ of $J_q(V)$. Solutions of $\mathcal{R}_q$ are its (local) sections.

We will only consider linear problems in the present paper, so $\mathcal{R}_q$ will be a vector bundle. Suppose V^0 and V^1 are two (vector) bundles. A linear qth order differential operator A can be thought of as a linear map $\mathcal{S}(V^0) \to \mathcal{S}(V^1)$. Then we can associate to A a bundle map $\mathcal{A} : J_q(V^0) \to V^1$ by the formula $A = \mathcal{A} j^q$. Now with $\mathcal{A}$ one can represent a differential equation as a zero set of a bundle map, $\mathcal{R}_q = \ker \mathcal{A}$, or $\mathcal{A}(x, j^q s(x)) = 0$.

Definition 2.2. The differential operator $j^r A : \mathcal{S}(V^0) \to \mathcal{S}(J_r(V^1))$ is said to be the r th prolongation of A. The associated morphism is denoted by $\mathcal{A}_r$.

Then we can define the prolongation of $\mathcal{R}_q$ by $\mathcal{R}_{q+r} = \ker \mathcal{A}_r$. We also define $\mathcal{R}^{(s)}_{q+r} = \pi^{q+r+s}_{q+r}(\mathcal{R}_{q+r+s})$. Note that $\mathcal{R}^{(s)}_{q+r} \subset \mathcal{R}_{q+r}$, but in general these sets are not equal.

Definition 2.3. A differential operator A is called sufficiently regular if $\mathcal{R}^{(s)}_{q+r}$ is a vector bundle for all $r \geq 0$ and $s \geq 0$.

If $\mathcal{M} \subset \mathbb{R}^n$ and the operator A has constant coefficients, then A is sufficiently regular.

Definition 2.4. A differential operator A (of order q) is called formally integrable if A is sufficiently regular and $\mathcal{R}^{(1)}_{q+r} = \mathcal{R}_{q+r}$ for all $r \geq 0$.

The formal integrability of an operator A of order q means that for any $r \geq 1$, all the differential consequences of order $q+r$ of the relations $As = 0$ may be obtained by means of differentiations of order no greater than r, and application of linear algebra.

The formal integrability cannot in general be checked in practice because there is an infinite number of conditions. Hence we need a stronger property, the involutivity of the system, which implies formal integrability, and can be checked in a finite number of steps. For the actual definition of involutivity we refer to [Spe69], [Pom78], [Sei01], and elsewhere. There is the following important result.

Theorem 2.1. *For a given sufficiently regular system $\mathcal{R}_q$ there are numbers r and s such that $\mathcal{R}^{(s)}_{q+r}$ is involutive,*

In practice to complete a system to the involutive form one may use DETools package [BHS01] in computer algebra system MuPAD [GOPW00].

The formal theory gives the notion of a principal symbol of the system which actually coincides with the classical concept.

Definition 2.5. Let $\mathcal{R}_q \subset J_q(V^0)$ be a sufficiently regular differential equation given by $\mathcal{R}_q = \ker \mathcal{A}$. By the principal symbol $\sigma^q(A)$ of A is meant the map $S^q(T^*\mathcal{M}) \otimes V^0 \to V^1$ defined by $\sigma^q(A) = \mathcal{A}\varepsilon_q$.

Consider a coordinate system on $\mathcal{M}$. Then a linear q th order partial differential equation $\mathcal{R}_q$ is given by

$$\begin{aligned} Au &:= \sum_{|\alpha| \leq q} A_\alpha(x) D^\alpha u \\ &= f, \end{aligned}$$

where $x \in O$, an open subset of $\mathbb{R}^n$, and $A_\alpha(x)$ is an $(l \times k)$-matrix of functions on O. Fixing any one form ξ we get a bundle map $\sigma^q(A)(\xi) : V^0 \to V^1$ which in coordinates is given by

$$\sigma^q(A)(\xi) = \sum_{|\alpha| = q} A_\alpha(x) \xi^\alpha.$$

To study the Fredholm property of overdetermined operators it is convenient to reduce the operator to a certain equivalent canonical form.

Definition 2.6. A differential operator $A : \mathcal{S}(V^0) \to \mathcal{S}(V^1)$ is called normalised if 1) A is a first order operator; 2) A is involutive; and 3) the principal symbol $\sigma^1(A) : T^*\mathcal{M} \otimes V^0 \to V^1$ is surjective.

The condition 3) means that there are no (explicit or implicit) algebraic (i.e., non-differential) relations between unknown functions in the system. If such relations exist, then we may use them to reduce the number of unknown functions.

Theorem 2.2. *Every sufficiently regular operator A can be transformed in a finite number of steps into an equivalent normalised operator.*

To consider boundary value problems we choose two bundles on the boundary $\mathcal{Y}$ of $\mathcal{M}$, i.e., $W^0, W^1 \to \mathcal{Y}$. The bundle $V^i\,|_{\mathcal{Y}} \to \mathcal{Y}$ is the restriction of $V^i \to \mathcal{M}$ to the boundary. If u is a section of $V^i \to \mathcal{M}$, then γu is the corresponding section of $V^i\,|_{\mathcal{Y}} \to \mathcal{Y}$. This map γ is called the trace map.

Definition 2.7. An operator $\Psi : \mathcal{S}(V^0) \times \mathcal{S}(W^0) \to \mathcal{S}(V^1) \times \mathcal{S}(W^1)$ of the form

$$\Psi(u, w) = \begin{pmatrix} \Psi_{1,1} & 0 \\ \gamma \Psi_{2,1} & \Psi_{2,2} \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix},$$

where $\Psi_{i,j}$ are differential operators, is called a boundary problem operator.

If $W^0 = 0$, we obtain in this way an operator $\Psi(u) = (Au, \gamma Bu)$ which defines a classical boundary problem on $\mathcal{M}$.

Definition 2.8. A boundary problem operator Ψ is said to be normalised if $\Psi_{1,1}$ is normalised and $\gamma \Psi_{2,1}$ contains only differentiation in directions tangent to the boundary.

Theorem 2.3. *Every boundary problem operator Ψ whose component $\Psi_{1,1}$ is sufficiently regular is equivalent to a normalised boundary problem operator.*

Note that this theorem still holds for the classical boundary value problems, i.e., in the case $\Psi_{2,2} = 0$.

We will not give the explicit mappings involved in the equivalence in Theorem 2.3 but we just indicate the steps of construction of the equivalent normalised boundary problem operator for a classical boundary problem (A, B). Namely, one should go through the following 4 steps:

1st step: Construct the involutive form of A.
2nd step: Prolong the system, if necessary, until the order of the system is higher than the order of normal derivatives in the boundary operator B.
3rd step: Construct an equivalent first order system.
4th step: Eliminate, if necessary, the extra variables (unknown functions) using the algebraic relations in the system.

It is worth pointing out that the system is involutive if and only if the equivalent first order system is involutive, cf. [Sei01].

We complete this section by constructing the normalised classical boundary problem for the familiar stationary Stokes problem in two dimensions.

Example 2.1. Consider the boundary problem in $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x_2 > 0\}$

$$\begin{array}{lrcll} A: & \left\{ \begin{array}{rcl} -\Delta u + \nabla p &=& f', \\ \operatorname{div} u &=& f_3 \end{array} \right. & & \text{in} & \mathcal{X} = \mathbb{R}^2_+, \\ B: & u &=& 0 \quad \text{on} & \mathcal{Y} = \partial \mathbb{R}^2_+, \end{array}$$

where $u = (u_1, u_2)$ is the velocity field, p is the pressure, and $f' = (f_1, f_2)$. By completing the above system to the involutive form we arrive at an overdetermined system

$$A^{(1)}: \quad \begin{cases} -\Delta u + \nabla p &= f', \\ \Delta p &= \operatorname{div} f' + \Delta f_3, \\ \operatorname{div} u &= f_3, \\ \partial_1^2 u_1 + \partial_1 \partial_2 u_2 &= \partial_1 f_3, \\ \partial_1 \partial_2 u_1 + \partial_2^2 u_2 &= \partial_2 f_3 \end{cases} \tag{2.1}$$

in $\mathcal{X}$. Introducing nine new variables (unknown functions)

$$\begin{array}{lll} v_{1,00} = u_1, & v_{1,10} = \partial_1 u_1, & v_{1,01} = \partial_2 u_1, \\ v_{2,00} = u_2, & v_{2,10} = \partial_1 u_2, & v_{2,01} = \partial_2 u_2, \\ v_{3,00} = p, & v_{3,10} = \partial_1 p, & v_{3,01} = \partial_1 p \end{array}$$

and substituting them into (2.1), and also adding the compatibility equations we get the first order system

$$A^{(2)}: \quad \begin{cases} -\partial_1 v_{1,10} - \partial_2 v_{1,01} + v_{3,10} &= f_1, \\ -\partial_1 v_{2,10} - \partial_2 v_{2,01} + v_{3,01} &= f_2, \\ \partial_1 v_{3,10} + \partial_2 v_{3,01} &= \operatorname{div} f' + \Delta f_3, \\ v_{1,10} + v_{2,01} &= f_3, \\ \partial_1 v_{1,10} + \partial_1 v_{2,01} &= \partial_1 f_3, \\ \partial_2 v_{1,10} + \partial_2 v_{2,01} &= \partial_2 f_3 \qquad \text{and} \\ \partial_1 v_{j,00} - v_{j,10} &= 0, \\ \partial_2 v_{j,00} - v_{j,01} &= 0, \\ \partial_2 v_{j,10} - \partial_1 v_{j,01} &= 0 \end{cases}$$

for $j = 1, 2, 3$. This system is not normalised since there is an algebraic relation $v_{1,10} + v_{2,01} = f_3$ between the dependent variables. Using this relation we can now eliminate the unknown function $v_{2.01}$ from the system and obtain the following normalised system

$$A^{(3)}: \quad \begin{cases} -\partial_1 v_{1,10} - \partial_2 v_{1,01} + v_{3,10} &= f_1, \\ \partial_2 v_{1,10} - \partial_1 v_{2,10} + v_{3,01} &= f_2 + \partial_2 f_3, \\ \partial_1 v_{3,10} + \partial_2 v_{3,01} &= \operatorname{div} f' + \Delta f_3 \qquad \text{and} \\ \partial_1 v_{j,00} - v_{j,10} &= 0, \\ \partial_1 v_{2,00} - v_{2,10} &= 0, \\ \partial_2 v_{j,00} - v_{j,01} &= 0, \\ \partial_2 v_{2,00} + v_{1,10} &= f_3, \\ \partial_2 v_{j,10} - \partial_1 v_{j,01} &= 0, \\ \partial_1 v_{1,10} + \partial_2 v_{2,10} &= \partial_1 f_3 \end{cases}$$

for $j = 1, 3$. Finally, substituting the new unknown functions in the boundary conditions, we obtain

$$B^{(1)}: \quad \begin{cases} v_{1,00} &= 0, \\ v_{2,00} &= 0. \end{cases}$$

Hence it follows that the classical boundary problem operator $(A^{(3)}, B^{(1)})$ is normalised.

3. Boundary problems for overdetermined elliptic systems

Let $\mathcal{X}$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\mathcal{Y} = \partial\mathcal{X}$, and $A(x, D)$ an $(l \times k)$-matrix of scalar partial differential operators with C^∞ coefficients in a neighbourhood of $\overline{\mathcal{X}} = \mathcal{X} \cup \mathcal{Y}$. We assume that $l \geq k$, i.e., the inhomogeneous system $Au = f$ is overdetermined, and that $A(x, D)$ is given a Douglis-Nirenberg principal symbol structure (N, M), where $M = (M_1, \ldots, M_k)$ and $N = (N_1, \ldots, N_l)$ are tuples of integer numbers. This means that the order of the entry $a_{i,j}(x, D)$ in the matrix $A(x, D)$ does not exceed $N_i + M_j$. As usual, one assumes without loss of generality that $N_i \leq 0$ and $\max\{N_1, \ldots, N_l\} = 0$, and that $a_{i,j} \equiv 0$ if $N_i + M_j < 0$.

The integer $o = \max\{N_i + M_j\}$ is a scalar order of $A(x, D)$, and we assume that o is positive.

We assume that the operator $A(x, D)$ is Douglis-Nirenberg elliptic, i.e., the matrix $\sigma(A)(x, \xi)$ constituted of the principal homogeneous parts of $a_{i,j}(x, \xi)$ of order $N_i + M_j$ has maximal rank k for all $x \in \overline{\mathcal{X}}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. This matrix satisfies the homogeneity condition in ξ

$$\sigma(A)(x, \lambda\xi) = \tilde{\varkappa}_\lambda \, \sigma(A)(x, \xi) \, \varkappa_\lambda^{-1} \tag{3.1}$$

for all $\lambda > 0$, where

$$\varkappa_\lambda = \begin{pmatrix} \lambda^{-M_1} & & 0 \\ & \ddots & \\ 0 & & \lambda^{-M_k} \end{pmatrix}, \quad \tilde{\varkappa}_\lambda = \begin{pmatrix} \lambda^{N_1} & & 0 \\ & \ddots & \\ 0 & & \lambda^{N_l} \end{pmatrix}.$$

We will consider the operator A in Sobolev spaces $H^{s+M}(\mathcal{X}) \to H^{s-N}(\mathcal{X})$ for $s = 0, 1, \ldots$, where

$$\begin{array}{rcl} H^{s+M}(\mathcal{X}) & = & H^{s+M_1}(\mathcal{X}) \times \ldots \times H^{s+M_k}(\mathcal{X}), \\ H^{s-N}(\mathcal{X}) & = & H^{s-N_1}(\mathcal{X}) \times \ldots \times H^{s-N_l}(\mathcal{X}). \end{array}$$

To study the kernel and cokernel of the operator $A : H^{s+M}(\mathcal{X}) \to H^{s-N}(\mathcal{X})$ one introduces the vector space $\mathfrak{M}(y, \eta)$ constituted of all bounded solutions to the system of ordinary differential equations

$$\sigma(A)\Big(y, \eta + \nu(y)\, \frac{1}{\imath} \frac{d}{dt}\Big) u(y, \eta; t) = 0 \tag{3.2}$$

on the semiaxis $t > 0$, depending on the parameter $(y, \eta) \in T^*\mathcal{Y} \setminus \{0\}$, where $y \in \mathcal{Y}$, $\eta \in T^*_y\mathcal{Y}$ is non-zero, and $\nu(y)$ is the unit inner normal vector to the boundary at the point y.

If the space $\mathfrak{M}(y, \eta)$ is trivial for all $(y, \eta) \in T^*\mathcal{Y} \setminus \{0\}$, which is possible even for $l = k$ provided $|N| + |M| \equiv 0$, then the operator $A : H^{s+M}(\mathcal{X}) \to H^{s-N}(\mathcal{X})$ has closed range and his kernel is finite dimensional, cf. for instance [DN55, Sol71]. In particular, the equation

$$Au = f \tag{3.3}$$

is normally solvable.

In the general case the kernel of the operator $A : H^{s+M}(\mathcal{X}) \to H^{s-N}(\mathcal{X})$ is finite dimensional only if additional boundary conditions

$$B(x, D)u\,|_{\mathcal{Y}} = u_0 \tag{3.4}$$

are posed on the solutions of (3.3), where $B(x, D)$ is an $(m \times k)$-matrix of scalar partial differential operators with C^∞ coefficients in a neighbourhood of the boundary $\mathcal{Y}$. We assume that $B(x, D)$ is given a Douglis-Nirenberg principal symbol structure (O, M), too, where $O = (O_1, \ldots, O_m)$, and that $B(x, D)$ complements $A(x, D)$ in the sense that the space $\mathfrak{M}(y, \eta)$ does not contain any non-zero solution satisfying the condition

$$\sigma(B)\Big(y, \eta + \nu(y)\, \frac{1}{\imath} \frac{d}{dt}\Big) u\,(y, \eta; 0) = 0 \tag{3.5}$$

for all $(y, \eta) \in T^*\mathcal{Y} \setminus \{0\}$, cf. [Sol71].

This condition requires that $m \geq \max \dim \mathfrak{M}(y, \eta)$, where the maximum is over all points $(y, \eta) \in T^*\mathcal{Y} \setminus \{0\}$.

In the case where $l = k$, $|N| + |M| > 0$, and A is properly elliptic, the function $\dim \mathfrak{M}(y, \eta)$ is constant on $T^*\mathcal{Y} \setminus \{0\}$ and is actually equal to $1/2\,(|N| + |M|)$. Then, for $m = 1/2\,(|N| + |M|)$, our condition on $B(x, D)$ just amounts to the Shapiro-Lopatinskii condition, and the boundary problem (3.3), (3.4) is called elliptic. An elliptic boundary problem gives rise to a Fredholm operator

$$\mathcal{A} := \begin{pmatrix} A \\ T \end{pmatrix} : H^{s+M}(\mathcal{X}) \to \begin{array}{c} H^{s-N}(\mathcal{X}) \\ \oplus \\ H^{s-1/2-O}(\mathcal{Y}) \end{array} \tag{3.6}$$

for $s > \max\{O_1, \ldots, O_m\}$, where T stands for B followed by the restriction to the boundary $\mathcal{Y}$, cf. [AD62, Vol65, Sol71].

Solomyak [Sol63] showed a properly elliptic first order differential operator A with $|N| + |M| = 4$, for which it is impossible to find any pseudodifferential operator B with $m = 2$ complementing A. It follows that an elliptic system of differential equations fails in general to possess a boundary problem satisfying the Shapiro-Lopatinskii condition. As is known [AB64], this restriction on elliptic systems is of topological character.

However, if one admits boundary matrices $B(x, D)$ with the number of rows m greater than $\max \dim \mathfrak{M}(y, \eta)$, then it is always possible to find suitable boundary conditions complementing A.

Theorem 3.1. *For each elliptic operator A of principal symbol structure (N, M) there is a boundary operator B of principal symbol structure (O, M) which complements A.*

Proof. We first assume that $N = 0$ and M is the k-row with equal entries o. By the Green formula for A, cf. for instance [Tar95], the Cauchy data

$$(u, D_\nu u, \ldots, D_\nu^{o-1} u)\,|_{\mathcal{Y}}$$

obviously complement A, where by D_ν is meant differentiation in the direction of the inner normal vector to $\mathcal{Y}$. We can thus ask about the minimal number of boundary conditions complementing A.

In a small neighbourhood of the boundary $\mathcal{Y}$ we can write the operator B in the form

$$\sum_{j=0}^{o-1} B_j\, D_\nu^j,$$

where B_j are differential operators of order $o - j$ acting only in directions tangent to $\mathcal{Y}$.

Given a point $(y, \eta) \in T^*\mathcal{Y} \setminus \{0\}$, we choose a $(k \times Q)$-matrix $\varPhi(y, \eta; t)$ whose columns constitute a basis of the space $\mathfrak{M}(y, \eta)$. Then the matrix

$$\varDelta(y, \eta) = \begin{pmatrix} \varPhi(y, \eta; 0+) \\ D_t\, \varPhi(y, \eta; 0+) \\ \ldots \\ D_t^{o-1}\, \varPhi(y, \eta; 0+) \end{pmatrix} \tag{3.7}$$

has maximal rank $Q = \dim \mathfrak{M}(y, \eta)$ at the point (y, η). Let $M(y, \eta)$ be a Q-rowed submatrix of $\varDelta(y, \eta)$ of non-zero determinant. Since

$$B(y, \eta + \nu(y) \frac{1}{\imath} \frac{d}{dt})\, \varPhi(y, \eta; 0+) = (B_0(y, \eta), \ldots, B_{o-1}(y, \eta))\, \varDelta(y, \eta)$$

we can choose elements of the matrix $(B_0(y, \eta), \ldots, B_{o-1}(y, \eta))$ in such a manner that $(B_0(y, \eta), \ldots, B_{o-1}(y, \eta))\, \varDelta(y, \eta) = M(y, \eta)$ be fulfilled. Then the operator B complements A at the point (y, η), and therefore in some neighbourhood $U(y, \eta)$ of this point in $T^*\mathcal{Y} \setminus \{0\}$.

If $U(y, \eta) = T^*\mathcal{Y} \setminus \{0\}$ then B is a desired operator. Since each row of the operator B contains only one non-zero element, whose order we denote by δ_i, then, setting $O_i = \delta_i - o$, we get $\sigma(B) = B$.

If $U(y,\eta)$ does not coincide with $T^*\mathcal{Y}\setminus\{0\}$, then we find a cover of $T^*\mathcal{Y}\setminus\{0\}$ by such neighbourhoods U and choose a minimal finite subcover $\{U_1,\dots,U_I\}$. In this case we form an operator

$$B = \begin{pmatrix} B_1 \\ \dots \\ B_I \end{pmatrix} \tag{3.8}$$

of $(Q_i \times k)$-matrices B_i complementing A in U_i. Then we reduce the number of rows of this matrix by eliminating equivalent rows, and assign orders to the rows of B in such a way that $\sigma(B) = B$. This readily yields a boundary operator B with desired properties.

In the general case where A is an elliptic operator with principal symbol structure (N, M) and $M \neq (o,\dots,o)$ (k-tuple), the data

$$(u_j, D_\nu u_j, \dots, D_\nu^{m_j-1} u_j)\,|_{\mathcal{Y}},$$

for $j = 1,\dots,k$, complement A, u_j being the j th component of the vector-valued function u. Hence, substituting zero rows for those rows in the matrix (3.7) which correspond to the derivatives $D_\nu^k u_j$ for $m_j \le k \le o-1$, we get a matrix having maximal rank at the point (y,η). The rest of the proof runs as before. $\square$

If a boundary operator B with principal symbol structure (O, M) complements an elliptic operator A with principal symbol structure (N, M), then the boundary problem (3.6) has finite-dimensional kernel and closed range. The cokernel of the operator (3.6) can be of infinite dimension even in the case $l = k$, provided that $m > 1/2\,(|N| + |M|)$, cf. [Sol71]. The proof of this reduces to constructing a left regulariser of the operator (3.6). The crucial point of this construction consists in constructing a left regulariser of the symbol

$$\sigma(\mathcal{A}) = \begin{pmatrix} \sigma(A) \\ \sigma(T) \end{pmatrix}$$

with constant coefficients in the half-space. Since this theorem is of great importance in the sequel, we show a simpler construction of a left regulariser of the operator $\sigma(\mathcal{A})$ in the half-space. To this end we use a special method of solution of a linear system presented in [Sob74, Ch. 1].

4. Left regulariser

As usual, we write $\mathbb{R}^n_+$ for the half-space $\{x \in \mathbb{R}^n : x_n > 0\}$. Let u be a smooth function on $\mathbb{R}^n_+$ rapidly decreasing at infinity and satisfying

$$\left\{ \begin{array}{rcll} A(D)u(x) & = & f(x) & \text{for} \quad x_n > 0, \\ B(D)u(x) & = & u_0(x') & \text{for} \quad x_n = 0, \end{array} \right. \tag{4.1}$$

where $A(D)$ is an $(l \times k)$-matrix of scalar differential operators with constant coefficients and principal symbol structure (N, M) and $B(D)$ is an $(m \times k)$-matrix of scalar differential operators with constant coefficients and principal symbol structure (O, M), both operators being without lower terms. Suppose A is elliptic, and B complements A. For simplicity we first consider the case where A is a homogeneous elliptic operator of order o, i.e., $n_i = 0$ and $m_j = o$ for all i, j, and the scalar order of B is less than o, i.e., all $O_1,\dots,O_m$ are negative. Given any function u on $\mathbb{R}^n_+$ which is smooth up to the boundary, we extend u by 0 to all of $\mathbb{R}^n$, and denote this extension by e_+u.

Denote by $\mathcal{F}_{x\mapsto\xi} e_+ u$ and $\mathcal{F}_{x'\mapsto\xi'} u$ the Fourier transformations of such functions in x and x'. Then $\mathcal{F}_{x\mapsto\xi} e_+ D^\alpha u = \xi^\alpha \mathcal{F}_{x\mapsto\xi} e_+ u$ for all $\alpha = (\alpha_1,\dots,\alpha_n)$ with $\alpha_n = 0$, and

$$\begin{aligned}\mathcal{F}_{x\mapsto\xi}e_+D_nu &= \xi_n\mathcal{F}_{x\mapsto\xi}e_+u+\imath\mathcal{F}_{x'\mapsto\xi'}u\,|_{x_n=0+},\\ \mathcal{F}_{x\mapsto\xi}e_+D_n^ju &= \xi_n^j\mathcal{F}_{x\mapsto\xi}e_+u+\imath\sum_{\iota=0}^{j-1}\xi_n^{j-1-\iota}U_\iota(\xi'),\end{aligned} \tag{4.2}$$

where $U_\iota(\xi')=D_n^\iota\mathcal{F}_{x'\mapsto\xi'}u\,|_{x_n=0+}$.

Set $U=(U_0,\dots,U_{o-1})$ and

$$A(\xi)=\sum_{j=0}^{o}A_j(\xi')\xi_n^j,$$

where $A_j(\xi')$ are homogeneous matrix-valued polynomials of degree $o-j$ in ξ'. Applying the Fourier transform to (4.1) and taking into account (4.2), we readily get

$$A(\xi)\mathcal{F}_{x\mapsto\xi}e_+u+\imath\sum_{j=1}^{o}\sum_{\iota=0}^{j-1}A_j(\xi')\xi_n^{j-1-\iota}U_\iota(\xi')=\mathcal{F}_{x\mapsto\xi}e_+f. \tag{4.3}$$

Write the second term on the right-hand side of (4.3) in the form $\tilde{A}U$. Changing the order of summation we see that

$$\begin{aligned}\tilde{A}U &= \imath\sum_{\iota=0}^{o-1}\Big(A(\xi)-\sum_{j=0}^{\iota}A_j(\xi')\xi_n^j\Big)\xi_n^{-1-\iota}U_\iota(\xi')\\ &= \imath\sum_{\iota=0}^{o-1}\Big(\sum_{j=0}^{o-1-\iota}A_{j+1+\iota}(\xi')\xi_n^j\Big)U_\iota(\xi'),\end{aligned} \tag{4.4}$$

and so $\tilde{A}=(\tilde{A}_0,\dots,\tilde{A}_{o-1})$ can be specified as a block matrix constituted of

$$\tilde{A}_\iota=\sum_{j=0}^{o-1-\iota}A_{j+1+\iota}(\xi')\xi_n^j,$$

matrix-valued polynomials of degree $o-1-\iota$ in ξ_n. Since $A(D)$ is elliptic, the matrix $A(\xi)$ has a left inverse matrix $A_L^{-1}(\xi)$ for all $\xi\in\mathbb{R}^n\setminus\{0\}$. A familiar way to choose such a matrix is

$$A_L^{-1}(\xi)=((A(\xi))^*A(\xi))^{-1}(A(\xi))^* \tag{4.5}$$

where $A^*=\bar{A}^T$ is the adjoint matrix. Let $R(\xi)=E-A(\xi)A_L^{-1}(\xi)$ for $\xi\in\mathbb{R}^n\setminus\{0\}$, where E is the identity $(l\times l)$-matrix. Then the system (4.3) is solvable if and only if

$$R(\xi)\left(\mathcal{F}_{x\mapsto\xi}e_+f-\tilde{A}U\right)=0 \tag{4.6}$$

for all $\xi\in\mathbb{R}^n\setminus\{0\}$, as is easy to see.

This condition can be transformed, when one uses the equality

$$\begin{aligned}((A(\xi))^*A(\xi))^{-1} &= \frac{\operatorname{adj}(A(\xi))^*A(\xi)}{\det(A(\xi))^*A(\xi)}\\ &= \frac{C(\xi)}{p(\xi)}\end{aligned} \tag{4.7}$$

for $\xi\in\mathbb{R}^n\setminus\{0\}$, where $p(\xi)$ is a properly elliptic polynomial of order 2ϖ, with $o\le\varpi\le ko$, and $C(\xi)$ is a $(k\times k)$-matrix of polynomials of degree $2(\varpi-o)$ which is obtained after possible cancellation of common divisors. Then $pR=pE-ACA^*$ is a matrix-valued polynomial of degree 2ϖ. However, $pR\,\tilde{A}U$ is a polynomial in ξ_n of degree less than 2ϖ. Indeed, since $RA\equiv 0$, we get

$$pR\,\tilde{A}U=-\imath pR\sum_{\iota=0}^{o-1}\sum_{j=0}^{\iota}A_j(\xi')\xi_n^{j-1-\iota}U_\iota(\xi')$$

which is due to (4.4). As $j \le \iota$, the desired conclusion is obvious. Hence it follows that

$$pR\,\tilde{A}U = \sum_{j=1}^{2\varpi} \xi_n^{j-1} P_j(\xi')U(\xi'). \tag{4.8}$$

The coefficients P_jU of this polynomial can be expressed through f. Indeed, multiplying equality (4.6) by ξ_n^{i-1} and integrating the product over a contour γ_+ enclosing all roots of the polynomial $p(\xi', \xi_n)$ in ξ_n lying in the upper half-plane, for a fixed ξ', we obtain

$$\sum_{j=1}^{2\varpi} \int_{\gamma_+} \frac{\xi_n^{i+j-2}}{p(\xi', \xi_n)}\, d\xi_n\, P_j(\xi')U(\xi'). = \int_{\gamma_+} \xi_n^{i-1} R(\xi) \mathcal{F}_{x\mapsto\xi} e_+ f\, d\xi_n \tag{4.9}$$

for $i = 1, \ldots, \varpi$. The matrix $M(\xi') = (M_{i,j}(\xi'))_{\substack{i=1,\ldots,\varpi \\ j=1,\ldots,\varpi}}$ with entries

$$M_{i,j}(\xi') = \int_{\gamma_+} \frac{\xi_n^{i+j-2}}{p(\xi', \xi_n)}\, d\xi_n$$

is non-degenerate for all $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$, this latter can be seen by verifying the Shapiro-Lopatinskii condition for the Dirichlet problem on $\mathbb{R}^n_+$ for the elliptic operator $p(D)$. Hence, setting $P_jU = 0$ for $j > \varpi$, we determine the other P_jU by inverting the matrix $M(\xi')$. The solution of (4.9) obtained in this way we can write in the form

$$\begin{aligned} PU &= \begin{pmatrix} P_1 \\ \vdots \\ P_{2\varpi} \end{pmatrix} U \\ &= \begin{pmatrix} T\,\mathcal{F}_{x\mapsto\xi} e_+ f \\ 0 \end{pmatrix} \end{aligned} \tag{4.10}$$

for $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$. Note that for $f = 0$ the conditions (4.6) and (4.10) are actually equivalent.

If (4.6) is satisfied, then

$$\mathcal{F}_{x\mapsto\xi} e_+ u = A_L^{-1}(\xi)\, (\mathcal{F}_{x\mapsto\xi} e_+ f - \tilde{A}U)$$

for $\xi \in \mathbb{R}^n \setminus \{0\}$, as is easy to check. Applying the inverse Fourier transform in ξ_n to both sides of this equality, we get

$$\mathcal{F}_{x'\mapsto\xi'} u(x', x_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\imath x_n \xi_n} A_L^{-1}(\xi)\, (\mathcal{F}_{x\mapsto\xi} e_+ f - \tilde{A}U)\, d\xi_n, \tag{4.11}$$

where the integral over the real axis can be replaced by the integral over γ_+. Furthermore, applying the Fourier transform in x' to the boundary condition in (4.1) yields

$$\begin{aligned} QU &:= \int_{\gamma_+} B(\xi) A_L^{-1}(\xi)\, \tilde{A}(\xi) U(\xi')\, d\xi_n \\ &= \int_{\gamma_+} B(\xi) A_L^{-1}(\xi) \mathcal{F}_{x\mapsto\xi} e_+ f\, d\xi_n - \mathcal{F}_{x'\mapsto\xi'} u_0, \end{aligned} \tag{4.12}$$

we have used the above formula for $\mathcal{F}_{x'\mapsto\xi'} u$. On the other hand, writing the boundary operator $B(D)$ in the form

$$\sum_{\iota=0}^{o-1} B_\iota(D') D_n^\iota$$

and introducing a block matrix $\tilde{B} = (B_0, \ldots, B_{o-1})$, we obtain from the boundary condition similarly to (4.3)

$$\tilde{B}(\xi')U(\xi') = \mathcal{F}_{x'\mapsto\xi'} u_0. \tag{4.13}$$

Lemma 4.1. *If the operator A is elliptic and B complements A, then the homogeneous system (4.10), (4.12), (4.13) (i.e., that corresponding to $f = 0$ and $u_0 = 0$) has only trivial solution $U = 0$ for all $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$.*

Proof. Let $U = (U_0, \ldots, U_{o-1})$ be a solution of the homogeneous system (4.10), (4.12), (4.13). Our objective will be to show that for each $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$ the vector-valued function

$$v(x_n) := -\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\imath x_n \xi_n} A_L^{-1}(\xi)\, \tilde{A}(\xi) U(\xi')\, d\xi_n$$

is a bounded solution of the equation

$$A\Big(\xi', \frac{1}{\imath}\frac{\partial}{\partial x_n}\Big) v(x_n) = 0 \tag{4.14}$$

for $x_n \neq 0$, satisfying the boundary conditions

$$B\Big(\xi', \frac{1}{\imath}\frac{\partial}{\partial x_n}\Big) v(x_n) = 0 \tag{4.15}$$

for $x_n = 0\pm$. Moreover, the jump of the function $D_n^\iota v(x_n)$ at $x_n = 0$ just amounts to

$$D_n^\iota v(0+) - D_n^\iota v(0-) = U_\iota \tag{4.16}$$

whenever $\iota = 0, 1, \ldots, o-1$.

To this end, pick contours γ_+ and γ_- which bound half-disks in the upper and lower half-planes containing all singularities of the matrix $((A(\xi))^* A(\xi))^{-1}$, i.e., all roots of the polynomial $p(\xi', \xi_n)$. Then the integral over the real axis in the formula for $v(x_n)$ can be replaced by the integral over γ_+ for $x_n > 0$ or over γ_- for $x_n < 0$. Hence it follows that $v(x_n)$ is bounded for $x_n \neq 0$. Applying the operator $A(\xi', D_n)$ to v readily get

$$A\Big(\xi', \frac{1}{\imath}\frac{\partial}{\partial x_n}\Big) v(x_n) = \frac{1}{2\pi} \int_{\gamma_\pm} e^{\imath x_n \xi_n} R\, \tilde{A}(\xi) U(\xi')\, d\xi_n - \frac{1}{2\pi} \int_{\gamma_\pm} e^{\imath x_n \xi_n} \tilde{A}(\xi) U(\xi')\, d\xi_n \tag{4.17}$$

for $x_n \neq 0$.

From the equality (4.10) with $f = 0$ we deduce that $R\, \tilde{A}(\xi) U(\xi') \equiv 0$ holds on the real axis. Therefore, the first integral on the right-hand side of (4.17) reduces to the integral over a semicircle $S_\pm(R) = \{\xi_n \in \mathbb{C} : |\xi_n| = R,\ \pm\Im \xi_n \geq 0\}$, which is actually independent of R large enough, and by (4.8) it tends to zero when $R \to \infty$. It follows that this integral is equal to zero. By the Cauchy theorem, the second integral on the right-hand side of (4.17) vanishes, too. This shows that (4.14) is fulfilled.

We now prove that $v(x_n)$ satisfies boundary conditions (4.15). For $x_n = 0+$ this follows immediately from (4.12) with $f = 0$ and $u_0 = 0$, because

$$\begin{aligned} B\Big(\xi', \frac{1}{\imath}\frac{\partial}{\partial x_n}\Big) v\ (0+) &= -\frac{1}{2\pi} \int_{\gamma_+} B(\xi)\, A_L^{-1}(\xi) \tilde{A}(\xi) U(\xi')\, d\xi_n \\ &= 0. \end{aligned}$$

To derive the boundary condition (4.15) at $x_n = 0-$ we make use of the last equality, obtaining

$$\begin{aligned} B\Big(\xi', \frac{1}{\imath}\frac{\partial}{\partial x_n}\Big) v\ (0-) &= -\frac{1}{2\pi} \int_{\gamma_-} B(\xi)\, A_L^{-1}(\xi) \tilde{A}(\xi) U(\xi')\, d\xi_n \\ &= \frac{1}{2\pi} \int_{S(R)} B(\xi)\, A_L^{-1}(\xi) \tilde{A}(\xi) U(\xi')\, d\xi_n \end{aligned}$$

where $S(R)$ is the circle of radius R around 0. Substituting (4.4) into this equality we get

$$\begin{aligned} B\Big(\xi', \frac{1}{\imath}\frac{\partial}{\partial x_n}\Big) v\ (0-) &= -\sum_{j=0}^{o-1}\sum_{\iota=0}^{o-1}\Big(\frac{1}{2\pi\imath}\int_{S(R)} \xi_n^{j-\iota-1}\, d\xi_n\Big) B_j(\xi')U_\iota(\xi') \\ &+ \sum_{\iota=0}^{o-1}\sum_{j=0}^{\iota}\Big(\frac{1}{2\pi\imath}\int_{S(R)} B(\xi)A_L^{-1}(\xi)\xi_n^{j-\iota-1}\, d\xi_n\Big) A_j(\xi')U_\iota(\xi'). \end{aligned} \tag{4.18}$$

The second integral on the right-hand side vanishes since the integrand is $O(|\xi_n|^{-2})$ as $\xi_n \to \infty$. It follows that

$$\begin{aligned} B\Big(\xi', \frac{1}{\imath}\frac{\partial}{\partial x_n}\Big) v\ (0-) &= -\sum_{j=0}^{o-1}\sum_{\iota=0}^{o-1} \delta_{\iota,j} B_j(\xi')U_\iota(\xi') \\ &= -\tilde{B}U, \end{aligned}$$

which vanishes by (4.13).

It remains to establish jump formulas (4.16). For this purpose we start with an equality

$$\begin{aligned} & D_n^k v(0+) - D_n^k v(0-) \\ &= -\frac{1}{2\pi}\int_{\gamma_+} \xi_n^k A_L^{-1}(\xi)\, \tilde{A}(\xi)U(\xi')\, d\xi_n + \frac{1}{2\pi}\int_{\gamma_-} \xi_n^k A_L^{-1}(\xi)\, \tilde{A}(\xi)U(\xi')\, d\xi_n \\ &= -\frac{1}{2\pi}\int_{S(R)} \xi_n^k A_L^{-1}(\xi)\, \tilde{A}(\xi)U(\xi')\, d\xi_n. \end{aligned}$$

Once again using equality (4.4) we split the integral on the right-hand side into the sum

$$\sum_{\iota=0}^{o-1}\Big(\frac{1}{2\pi\imath}\int_{S(R)} \xi_n^{k-\iota-1} d\xi_n\Big) U_\iota(\xi') - \sum_{\iota=0}^{o-1}\sum_{j=0}^{\iota}\Big(\frac{1}{2\pi\imath}\int_{S(R)} A_L^{-1}(\xi)\xi_n^{k+j-\iota-1} d\xi_n\Big) A_j(\xi')U_\iota(\xi'). \tag{4.19}$$

The last integral here vanishes if $k < o$. Hence it follows that

$$\begin{aligned} D_n^k v(0+) - D_n^k v(0-) &= \sum_{\iota=0}^{o-1} \delta_{k,\iota} U_\iota \\ &= U_k \end{aligned}$$

whenever $k = 0, 1, \ldots, o-1$, as desired.

Since the boundary operator B complements A, we conclude that $v(x_n) = 0$ for all $x_n > 0$, and so $v(x_n) = 0$ holds for all $x_n < 0$, too. Indeed, to each non-trivial solution $v(\xi', x_n)$ of the problem (4.14), (4.15) on the semiaxis $\mathbb{R}_{\leq 0}$ there corresponds the solution $u(\xi', x_n) := (-1)^o v(-\xi', -x_n)$ of the problem (4.14), (4.15) on the semiaxis $\mathbb{R}_{\geq 0}$, and this latter has only trivial solution. Using the jump formulas (4.16) we thus conclude that $U(\xi') = 0$ for all $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$, which completes the proof. □

The principal significance of Lemma 4.1 is in the assertion that the matrix

$$\mathcal{B}(\xi') = \begin{pmatrix} P(\xi') \\ Q(\xi') \\ \tilde{B}(\xi') \end{pmatrix}$$

of the system (4.10), (4.12), (4.13) has a left inverse for all $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$. Denote it by $\mathcal{B}_L^{-1}(\xi')$, and the right-hand side of the system by $\mathit{\Delta}(\mathcal{F}_{x\mapsto\xi} e_+ f, \mathcal{F}_{x'\mapsto\xi'} u_0)$.

Then we obtain

$$U = \mathcal{B}_L^{-1}(\xi')\, \Delta(\mathcal{F}_{x\mapsto\xi} e_+ f, \mathcal{F}_{x'\mapsto\xi'} u_0). \tag{4.20}$$

By the very construction, if $f = Au$ and $u_0 = Bu\,|_{\mathcal{Y}}$ for some $u(x)$, then (4.20) gives

$$U(\xi') = \Big(\mathcal{F}_{x'\mapsto\xi'} u(x', 0+), \ldots, D_n^{o-1} \mathcal{F}_{x'\mapsto\xi'} u(x', 0+)\Big).$$

Substituting (4.20) into (4.11) yields

$$\begin{aligned} & \mathcal{F}_{x'\mapsto\xi'} u(x', x_n) \\ = \;& \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\imath x_n \xi_n} A_L^{-1}(\xi)\, (\mathcal{F}_{x\mapsto\xi} e_+ f - \tilde{A}(\xi) \mathcal{B}_L^{-1}(\xi') \Delta(\mathcal{F}_{x\mapsto\xi} e_+ f, \mathcal{F}_{x'\mapsto\xi'} u_0))\, d\xi_n. \end{aligned} \tag{4.21}$$

Applying the inverse Fourier transform in ξ', one can now derive a formula for the solution of problem (4.1) in the form of convolution of f and u_0 with matrices of Poisson kernels, cf. [Sol71].

We are however interested in constructing a mere left regulariser of problem (4.1). To this end we multiply the integrand in (4.21) by a non-negative function $\chi(\xi)$ of class $C^\infty(\mathbb{R}^n)$ which is equal to 1 for $|\xi| > 1$ and 0 for $|\xi| < 1/2$. Such functions are called excision functions. Applying now the inverse Fourier transform in ξ', we get an operator

$$\begin{aligned} & \varPi \begin{pmatrix} f \\ u_0 \end{pmatrix} \\ = & \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\imath\langle x,\xi\rangle} \chi(\xi) A_L^{-1}(\xi) \Big(\mathcal{F}_{x\mapsto\xi} e_+ f - \tilde{A}(\xi) \mathcal{B}_L^{-1}(\xi') \Delta(\mathcal{F}_{x\mapsto\xi} e_+ f, \mathcal{F}_{x'\mapsto\xi'} u_0)\Big) d\xi \end{aligned} \tag{4.22}$$

which is a left regulariser of problem (4.1). Indeed, using (4.3) and (4.20), we readily get

$$\begin{aligned} & \varPi \begin{pmatrix} Au \\ Bu\,|_{\mathcal{Y}} \end{pmatrix} \\ = \;& \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\imath\langle x,\xi\rangle} \chi(\xi) A_L^{-1}(\xi) \Big(A(\xi) \mathcal{F}_{x\mapsto\xi} e_+ u + \tilde{A}(\xi) U(\xi') - \tilde{A}(\xi) U(\xi')\Big) d\xi \\ = \;& e_+ u - S(e_+ u) \end{aligned}$$

where

$$S(e_+ u)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\imath\langle x,\xi\rangle} (1 - \chi(\xi))\, \mathcal{F}_{x\mapsto\xi} e_+ u\, d\xi$$

is a pseudodifferential operator of order $-\infty$ in $\mathbb{R}^n$, as desired.

We now discuss those modifications in the construction of a left regulariser which should be done in the general case where A is an elliptic operator of principal symbol structure (N, M). Formula (4.22) for a left regulariser and other main formulas remain still valid. Note that we can assume without loss of generality that the tuple N is equal to zero. For if $N_i < 0$, one can raise the order of the i th row in the operator A by applying the derivatives D^α with $|\alpha| = -N_i$ to this row, cf. [Sol71]. In other words, one can find an elliptic differential operator $\varLambda$ with constant coefficients and principal symbol structure $(0, -N)$, such that $\varLambda$ has finite-dimensional kernel, the composition $\varLambda A$ is of principal symbol structure $(0, M)$, and B complements $\varLambda A$. Then the composition $\varPi\varLambda$ of the operator $\varLambda$ and a left regulariser $\varPi$ of the problem

$$\begin{pmatrix} \varLambda A \\ T \end{pmatrix}$$

is a left regulariser of the genuine boundary problem $\mathcal{A}$.

Thus, let $N = 0$ and $o = \max\{M_1, \dots, M_k\}$ be the scalar order of the operator A. The decomposition

$$A(\xi) = \sum_{j=0}^{o} A_j(\xi')\xi_n^j$$

remains still valid, however, $A_j(\xi')$ are possibly inhomogeneous matrix-valued polynomials in ξ' of order $\le o - j$. Moreover, some $A_j(\xi')$ may vanish identically. For the components $a_{i,j}(\xi)$ of the matrix $A(\xi)$ one can write exacter equalities

$$a_{i,j}(\xi) = \sum_{\mu=0}^{M_j} a_{i,j;\mu}(\xi')\xi_n^\mu.$$

Analogously, writing (4.4) componentwise, we obtain

$$(\tilde{A}U)_i = \imath \sum_{j=1}^{k} \sum_{\iota=0}^{M_j-1} \sum_{\mu=0}^{M_j-1-\iota} a_{i,j;\iota+\mu+1}(\xi')\xi_n^\mu U_{\iota,j}(\xi'),$$

which shows that $\tilde{A}U$ is completely determined by the components $U_{0,j}, \dots, U_{M_j-1,j}$ of the vector U, for $j = 1, \dots, k$. Write U' for these components, then $\tilde{A}U = 0$ if $U' = 0$.

The operators A_L^{-1} and R have principal symbol structures $(-M, 0)$ and $(0, 0)$, respectively. The matrix A^*A has principal symbol structure (M, M), and so $\det A^*(\xi)A(\xi)$ is a homogeneous polynomial in ξ of order $2|M|$. The polynomial $p(\xi)$ of the representation (4.7) has order 2ϖ, and $\min\{M_1, \dots, M_k\} \le \varpi \le |M|$. The product pR is still a matrix-valued polynomial of degree 2ϖ, however, the degree of the polynomial $pR\tilde{A}U$ in ξ_n is less than 2ϖ. Hence it follows that $R\tilde{A}U \to 0$ as $\xi_n \to \infty$.

The boundary operator B bears principal symbol structure (O, M). The components $O_1, \dots, O_m$ of the tuple O are negative, if we assume that the orders of the elements $b_{i,j}$ of the matrix B do not exceed M_j. Writing

$$b_{i,j}(\xi) = \sum_{\mu=0}^{O_i+M_j} b_{i,j;\mu}(\xi')\xi_n^\mu$$

we readily observe that similarly to $\tilde{A}U$ the vector $\tilde{B}U$ is completely determined by the part U' of the vector U. Therefore, the relations (4.10), (4.12), (4.13) can be thought of as a system relative to the unknown vector U'.

Then Lemma 4.1 remains valid. This follows by the same method as above, the only difference being in replacing the jump formulas (4.16) by

$$D_n^\iota v_j(0+) - D_n^\iota v_j(0-) = U_{\iota,j} \tag{4.23}$$

for every $j = 1, \dots, k$ and for every component $U_{\iota,j}$ of the vector U_ι, where $\iota = 0, 1, \dots, M_j - 1$.

In order to establish (4.18) and (4.23), we rewrite (4.18) and (4.19) componentwise and take into account that the operators BA_L^{-1} and A_L^{-1} have principal symbol structures $(O, 0)$ and $(-M, 0)$. Then the last integrals in (4.18) and (4.19) are still equal to zero, for $O_i < 0$ and $\varpi < M_j$. Furthermore, if $v(\xi', x_n)$ is a solution to the problem (4.14), (4.15) on the semiaxis $\mathbb{R}_-$, then by the suitable homogeneity of the operators A and B the vector-valued function $u(\xi', x_n) := \varkappa_{-1}^{-1} v(-\xi', -x_n)$ is a solution of the problem (4.14), (4.15) on the semiaxis $\mathbb{R}_+$. It follows that the problem in $\mathbb{R}_-$ has also a mere trivial solution. By (4.23) we conclude that $U' = 0$ for all $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$, as desired.

5. Boundary integral equations

Let A and B be $(k \times k)$- and $(m \times k)$-matrices of scalar partial differential operators with principal symbol structures (N, M) and (O, M) and C^∞ coefficients in the closure of $\mathcal{X}$, respectively. We moreover assume that A is elliptic in $\overline{\mathcal{X}}$, and $m = (|N| + |M|)/2$.

Consider the general boundary problem

$$\left\{ \begin{array}{rclll} A(x,D)u(x) & = & f(x) & \text{for} & x \in \mathcal{X}, \\ B(x,D)u(x) & = & u_0(x) & \text{for} & x \in \mathcal{Y}. \end{array} \right. \tag{5.1}$$

We first study a homogeneous problem, i.e., that with $f = 0$. To this end, we consider also an auxiliary elliptic boundary problem for solutions of the system $Au = 0$,

$$\left\{ \begin{array}{rclll} A(x,D)u(x) & = & 0 & \text{for} & x \in \mathcal{X}, \\ C(x,D)u(x) & = & u_0(x) & \text{for} & x \in \mathcal{Y}, \end{array} \right. \tag{5.2}$$

where C is an $(m \times k)$-matrix of scalar partial differential operators with principal symbol structures (P, M) and smooth coefficients. To the elliptic problem (5.2) there corresponds an operator

$$C_{\mathcal{Y}} : H^{s+M}(\mathcal{X}) \cap \ker A \to H^{s-P-1/2}(\mathcal{Y}) \tag{5.3}$$

defined by $C_{\mathcal{Y}} u := Cu\,|_{\mathcal{Y}}$. It possesses both left and right regulariser Π, that is $\Pi = C^{-1}_{\mathcal{Y},L} = C^{-1}_{\mathcal{Y},R}$, where

$$\begin{array}{rclll} C^{-1}_{\mathcal{Y},L} C_{\mathcal{Y}} u & = & u - S_0 u & \text{if} & Au = 0; \\ C_{\mathcal{Y}} C^{-1}_{\mathcal{Y},R} v & = & v - S_1 v & \text{and} & AC^{-1}_{\mathcal{Y},R} v = 0, \end{array} \tag{5.4}$$

where S_0 and S_1 are smoothing operators of order $-\infty$.

We say that the operator (5.3) has principal symbol structure $(P + 1/2, M)$. Then the operator Π has principal symbol structure $(-M, -P - 1/2)$.

In what follows we need some results of [VG67, P. 2]. Hence we shortly present them with necessary complements. Consider the operator

$$\Psi : H^{s-P-1/2}(\mathcal{Y}) \to H^{s-O-1/2}(\mathcal{Y}) \tag{5.5}$$

given by $\Psi v := B_{\mathcal{Y}} C^{-1}_{\mathcal{Y},L} v$. It has principal symbol structure $(O + 1/2, -P - 1/2)$, what just amounts to $(O, -P)$. The operator Ψ is well known to be pseudodifferential, and its complete symbol $\sigma(\Psi)$ can be computed explicitly. Note that the operator Ψ is elliptic if and only if the boundary problem (5.1) is elliptic. It is important that existence and smoothness theorems for problem (5.1) are encoded in the operator (5.5).

More precisely, let H and H_1, H_2 be Banach spaces, $H \hookrightarrow \big(\cup_s H^s(\mathcal{X})^k\big) \cap \ker A$ and $H_1, H_2 \hookrightarrow \cup_s H^s(\mathcal{Y})^m$. Assume that these embeddings are continuous, when the target space is endowed with the topology of distributions, and that H_2 contains $C^\infty(\mathcal{Y})^m$. If in the diagram

$$\begin{tikzcd} H \arrow[r, "B_{\mathcal{Y}}"] \arrow[d, "C_{\mathcal{Y}}"'] & H_2 \\ H_1 \arrow[ur, "\Psi"'] & \end{tikzcd}$$

the operator $C_{\mathcal{Y}}$ is Fredholm, then for the operator $B_{\mathcal{Y}}$ to possess a regulariser (both left and right one) it is necessary and sufficient that Ψ would possess such a regulariser. Hence it follows that the operator $B_{\mathcal{Y}}$ is Fredholm if and only if Ψ is Fredholm.

As mentioned above, for some elliptic differential operators A it is still impossible to find a boundary operator C satisfying the Shapiro-Lopatinskii condition, and so, to choose a Fredholm operator $C_{\mathcal{Y}}$. However, in any case we can find, by Theorem 3.1, a boundary operator C which complements A, i.e., we may always choose an operator $C_{\mathcal{Y}}$ possessing a left regulariser $C^{-1}_{\mathcal{Y},L}$.

Theorem 5.1. *Let $C_{\mathcal{Y}}$ possess a left regulariser $C^{-1}_{\mathcal{Y},L}$. Then the operator $B_{\mathcal{Y}}$ has finite-dimensional kernel and is normally solvable, provided that Ψ possesses a left regulariser. If moreover $C_{\mathcal{Y}}$ possesses a right regulariser $C^{-1}_{\mathcal{Y},R}$, then $B_{\mathcal{Y}}$ is Fredholm if and only if so is Ψ.*

Proof. Let $C^{-1}_{\mathcal{Y},L}$ be a left regulariser of $C_{\mathcal{Y}}$. Then

$$\begin{aligned} B_{\mathcal{Y}} &= B_{\mathcal{Y}}\left(C^{-1}_{\mathcal{Y},L}C_{\mathcal{Y}} + S_0\right) \\ &= \Psi C_{\mathcal{Y}} + B_{\mathcal{Y}}S_0, \end{aligned} \tag{5.6}$$

where $B_{\mathcal{Y}}S_0$ is an operator of order $-\infty$ which defines a compact mapping of H to H_2. It follows that $B_{\mathcal{Y}}$ possesses a left regulariser $B^{-1}_{\mathcal{Y},L} = C^{-1}_{\mathcal{Y},L}\,\Psi^{-1}_{L}$, and therefore the operator $B_{\mathcal{Y}} : H \to H_2$ has finite-dimensional kernel and is normally solvable. If moreover both $C_{\mathcal{Y}}$ and Ψ possess right regularisers $C^{-1}_{\mathcal{Y},R}$ and Ψ^{-1}_{R}, then the operator $B_{\mathcal{Y}}$ possesses also a right regulariser $B^{-1}_{\mathcal{Y},R} = C^{-1}_{\mathcal{Y},R}\,\Psi^{-1}_{R}$, and so it is Fredholm. Conversely, if $C_{\mathcal{Y}}$ and $B_{\mathcal{Y}}$ are Fredholm, then Ψ is obviously Fredholm, too, as desired. □

From the equality (5.6) the results of [VG67] immediately follow. Yet another consequence of (5.6) is

Theorem 5.2. *If a smoothness theorem is valid for the operators $C_{\mathcal{Y}}$ and Ψ, then it also holds for the operator $B_{\mathcal{Y}}$, i.e., any solution of problem (5.1) with $f = 0$ in $\cup_s H^s(\mathcal{X})^k$ actually belongs to H if $u_0 \in H_2$.*

6. Evaluation of the symbol of Ψ

In this section we discuss the boundary system $\Psi v = g$ in detail. It is known, cf. [VG67], that in order to compute a complete symbol $\sigma(\Psi)$ of the system in a neighbourhood of a point $(y,\eta) \in T^*\mathcal{Y} \setminus \{0\}$ it suffices to use a local regulariser Π_Q of (5.1) instead of Π. By this is meant that

$$\begin{aligned} C_{\mathcal{Y}}\Pi_Q v &= v - S_{1,Q}v, \\ A\Pi_Q v &= -T_{1,Q}v \end{aligned}$$

for all vector-valued distributions v supported in a small neighbourhood U of y on $\mathcal{Y}$, where $S_{1,Q}$ and $T_{1,Q}$ are smoothing operators whose order tends to $-\infty$ when $Q \to \infty$. The operators Ψ and $B_{\mathcal{Y}}\Pi_Q$ differ by a smoothing operator S_Q of the same type, i.e., $\Psi v - B_{\mathcal{Y}}\Pi_Q v = S_Q v$ for all $\mathbb{C}^m$-valued distributions with support in U.

The construction of the operator Π_Q reduces to solving a boundary problem on the semiaxis $\mathbb{R}_+$ for a system of ordinary differential equations with parameter $(y,\eta) \in T^*\mathcal{Y} \setminus \{0\}$.

Let O be a domain in $\mathcal{X}$, such that $\overline{O} \cap \mathcal{Y} = \overline{U}$, where U is a sufficiently small neighbourhood of a point $y \in \mathcal{Y}$. In this domain we introduce coordinates (y,t) in which O is defined by the inequalities $|y|^2 + t^2 < \varepsilon^2$ and $t > 0$, for some $\varepsilon > 0$, while the boundary $\mathcal{Y}$ is given by the equality $t = 0$. The differential operator $A(x,D)$ written in these coordinates is denoted by $A(y,t,D_y,D_t)$, and its complete symbol by $A(y,t,\eta,\tau)$.

We now extend the twisted homogeneity (3.1) to include not only covariables η and τ but also the variables y and t, which will allow us to freeze coefficients. A matrix-valued function $F(z,t,\eta,\tau)$ is called twisted homogeneous of order (H,M) if

$$F(\lambda^{-1}z, \lambda^{-1}t, \lambda\eta, \lambda\tau) = \tilde{\varkappa}_\lambda\, F(z,t,\eta,\tau)\, \varkappa^{-1}_\lambda \tag{6.1}$$

for all $\lambda > 0$, where

$$\varkappa_\lambda = \begin{pmatrix} \lambda^{-M_1} & & 0 \\ & \ddots & \\ 0 & & \lambda^{-M_k} \end{pmatrix}, \qquad \tilde{\varkappa}_\lambda = \begin{pmatrix} \lambda^{H_1} & & 0 \\ & \ddots & \\ 0 & & \lambda^{H_l} \end{pmatrix}.$$

For each $y_0 \in U$ and $Q > 0$, the operators A, B and C can be written in the form

$$\begin{aligned} A(y,t,D_y,D_t) &= \sum_{q=0}^{Q} A_q(y_0, y-y_0, t, D_y, D_t) + A_{Q+1,R}(y_0, y-y_0, t, D_y, D_t), \\ B(y,t,D_y,D_t) &= \sum_{q=0}^{Q} B_q(y_0, y-y_0, t, D_y, D_t) + B_{Q+1,R}(y_0, y-y_0, t, D_y, D_t), \\ C(y,t,D_y,D_t) &= \sum_{q=0}^{Q} C_q(y_0, y-y_0, t, D_y, D_t) + C_{Q+1,R}(y_0, y-y_0, t, D_y, D_t), \end{aligned} \tag{6.2}$$

where $A_q(y_0, z, t, \eta, \tau)$, $B_q(y_0, z, t, \eta, \tau)$ and $C_q(y_0, z, t, \eta, \tau)$ are matrix-valued polynomials in z, t, η, τ, depending on the parameter y_0, which are moreover homogeneous of orders $(N-q, M)$, $(O-q, M)$ and $(P-q, M)$, respectively. The coefficients of the operators $A_{Q+1,R}$, $B_{Q+1,R}$ and $C_{Q+1,R}$ vanish at the point $(z,t) = (0,0)$ for Q large enough, and the multiplicity of this zero increases if $Q \to \infty$. The expansions (6.2) are obtained by applying the Taylor formula to each element of the matrices A, B and C around the point $(y,t) = (y_0, 0)$, and arranging into groups the terms which have the same generalised order of homogeneity. In particular, A_0, B_0 and C_0 are principal parts of the corresponding operators with coefficients freezing at the point $(y,t) = (y_0, 0)$.

Set $\tilde{A}_q = A_q(y_0, -D_\eta, t, \eta, D_t)$ and similarly for $\tilde{B}_q$, $\tilde{C}_q$. Consider a family of boundary problems for systems of ordinary differential equations on the semiaxis $t > 0$, parametrised by $(y_0, \eta) \in T^*U \setminus \{0\}$,

$$\begin{aligned} \tilde{A}_0\, \Sigma_0(y_0, \eta; t) &= 0, \\ \tilde{C}_0\, \Sigma_0(y_0, \eta; 0) &= E_m; \end{aligned}$$

for $q = 0$ and

$$\begin{aligned} \tilde{A}_0\, \Sigma_q(y_0, \eta; t) &= -\tilde{A}_1\, \Sigma_{q-1}(y_0, \eta; t) - \ldots - \tilde{A}_q\, \Sigma_0(y_0, \eta; t), \\ \tilde{C}_0\, \Sigma_q(y_0, \eta; 0) &= -\tilde{C}_1\, \Sigma_{q-1}(y_0, \eta; 0) - \ldots - \tilde{C}_q\, \Sigma_0(y_0, \eta; 0), \end{aligned} \tag{6.3}$$

for $q = 1, \ldots, Q$, where $\Sigma_q(y_0, \eta; t)$ are unknown $(k \times m)$-matrices. To guarantee the uniqueness we require $\Sigma_q(y_0, \eta; t)$ to vanish when $t \to +\infty$.

The problems (6.3) are uniquely solvable if and only if the boundary problem (5.1) is elliptic, i.e., satisfies the Shapiro-Lopatinskii condition. Furthermore, each matrix $\Sigma_q(y_0, \eta; t)$ is twisted homogeneous of order $(-M, -P-q)$ in (t, η) with C^∞ entries for $(y_0, \eta) \in T^*U \setminus \{0\}$.

We now define the operator

$$\mathrm{op}\,(\chi \Sigma_q)\, v\, (y,t) = \mathcal{F}^{-1}_{\eta \mapsto y} \mathcal{F}_{y \mapsto \eta} \chi(\eta) \Sigma_q(y, \eta; t) v(y)$$

which maps vector-valued distributions v with compact support in U to vector-valued distributions in O. Here $\chi(\eta)$ is an excision function on $\mathbb{R}^{n-1}$ which vanishes for $|\eta| < 1/2$ and is equal to 1 for $|\eta| > 1$. Then the family of operators

$$\Pi_Q = \sum_{q=0}^{Q} \mathrm{op}\,(\chi \Sigma_q)$$

is a local regulariser of problem (5.1).

From what has been said it readily follows that the complete symbol of the operator Ψ is given in local coordinates $y = (y_1, \ldots, y_{n-1})$ in the neighbourhood U by the formal series

$$\sigma(\Psi)(y,\eta) \sim \sum_{j=0}^{\infty} a_j(y,\eta), \tag{6.4}$$

with $a_j(y,\eta) = \sum_{p+q=j} \tilde{B}_p \Sigma_q\,(y,\eta;0)$.

In particular, the symbol of Ψ of order $(O,-P)$ is equal to $\tilde{B}_0(y,\eta,D_t)\Sigma_0\,(y,\eta;0)$, which implies that the problem (5.1) with $f = 0$ is elliptic if and only if so is the operator Ψ.

7. Generalised elliptic boundary problems

In [VG67] one discusses boundary problems which lead to uniformly non-elliptic systems on $\mathcal{Y}$. Developing [Sak80], we now consider boundary problems which lead to a more broad class of non-elliptic systems.

A boundary problem (5.1) with $f = 0$ is called generalised elliptic if the system $\Psi v = g$ is generalised elliptic, i.e., if there are admissible global transformations $T_1, \ldots, T_n$, such that the composition $T_n \ldots T_1 \Lambda^{-O}\,\Psi$ is elliptic in $\overline{\mathcal{Y}}$ and has order $(0,-P)$. Here, Λ^{-O} is the diagonal $(m \times m)$-matrix with entries $\delta_{i,j}\Lambda^{-O_j}$, where Λ is an arbitrary first order elliptic operator on $\mathcal{Y}$ with principal symbol $|\eta|$. See [Sak97] for more details.

The quantity $\delta(\Psi) + |P|$, where $\delta(\Psi)$ is the degree of non-ellipticity of the operator Ψ, is said to be the degree of non-ellipticity of boundary problem (5.1). The following lemma shows that this definition is correct.

Lemma 7.1. *The property of being generalised elliptic and the degree of non-ellipticity of problem (5.1) with $f = 0$ are independent of the choice of the auxiliary elliptic problem (5.2).*

Proof. Let $\mathfrak{C}_{\mathcal{Y}}$ be yet another operator with principal symbol structure (R,M) defining an elliptic boundary problem for the system $Au = 0$, and let $\mathfrak{P}$ be a regulariser of this problem. Then $\Psi' = B_{\mathcal{Y}}\mathfrak{P}$ and $\Psi'' = C_{\mathcal{Y}}\mathfrak{P}$ are matrix-valued pseudodifferential operators on $\mathcal{Y}$ with principal order structures $(O,-R)$ and $(P,-R)$, respectively, and the operator Ψ'' is elliptic. From (5.4) we obtain

$$\begin{aligned} \Psi' v &= B_{\mathcal{Y}} C_{\mathcal{Y},L}^{-1} C_{\mathcal{Y}} \mathfrak{P} v + B_{\mathcal{Y}} S_0 \mathfrak{P} v \\ &= \Psi \Psi'' v + S v, \end{aligned} \tag{7.1}$$

where S is a pseudodifferential operator of order $-\infty$. If the operator Ψ is generalised elliptic, i.e., if there are admissible global transformations $T_1, \ldots, T_n$, such that the composition $T_n \ldots T_1 \Lambda^{-O}\,\Psi$ is elliptic in $\overline{\mathcal{Y}}$ and has order $(0,-P)$, then the operator Ψ' is generalised elliptic, too, for by (7.1) the operator

$$T_n \ldots T_1 \Lambda^{-O}\,\Psi' v = \left(T_n \ldots T_1 \Lambda^{-O}\,\Psi\right)\,\Psi'' v + S' v$$

is elliptic in $\overline{\mathcal{Y}}$ and has order $(0,-R)$. Furthermore, if $\delta(\Psi)$ and $\delta(\Psi')$ are the degrees of non-ellipticity of Ψ and Ψ', then $\delta(\Psi') = \delta(\Psi) + |P| - |R|$, which just amounts to $\delta(\Psi') + |R| = \delta(\Psi) + |P|$. That is, the degree of non-ellipticity of problem (5.1) $\delta(\Psi) + |P|$ does not depend on the choice of the auxiliary problem (5.2), as desired. □

We now turn to function spaces in which the operator $B_{\mathcal{Y}}$ corresponding to the problem (5.1) with $f = 0$ acts. Recall, cf. [Sak78], that any finite sequence of elliptic operators $T_1, \ldots, T_n$ determines a space $H_T^{s-O-1/2}(\mathcal{Y})$ of generalised functions v on $\mathcal{Y}$, such that

$$T_n \ldots T_1 \Lambda^{-O} v \in H^{s-1/2}(\mathcal{Y}).$$

It is easy to verify that

$$H^{s-O-n-1/2}(\mathcal{Y}) \hookrightarrow H_T^{s-O-1/2}(\mathcal{Y}) \hookrightarrow H^{s-O-1/2}(\mathcal{Y}),$$

and the same space $H_T^{s-O-1/2}(\mathcal{Y})$ can be described by different equivalent finite sequences.

By Theorem 3 of [Sak78], to any generalised elliptic operator Ψ there corresponds a space $H_T^{s-O-1/2}(\mathcal{Y})$, determined by an arbitrary finite sequence reducing Ψ to an elliptic operator, such that

$$\Psi : H^{s-P-1/2}(\mathcal{Y}) \to H_T^{s-O-1/2}(\mathcal{Y}) \tag{7.2}$$

is a Fredholm operator.

Conversely, if there is a space $H_T^{s-O-1/2}(\mathcal{Y})$ such that the operator (7.2) is Fredholm, then Ψ is a generalised elliptic operator and each finite sequence of operators determining $H_T^{s-O-1/2}(\mathcal{Y})$ reduces Ψ to an elliptic operator of principal symbol structure $(0, -P)$.

Any generalised elliptic boundary problem gives rise to a Fredholm operator $B_{\mathcal{Y}}$ acting as

$$B_{\mathcal{Y}} : H^{s+M}(\mathcal{X}) \cap \ker A \to H_T^{s-O-1/2}(\mathcal{Y}), \tag{7.3}$$

the latter space being determined by an arbitrary finite sequence reducing Ψ to an elliptic operator. It is independent of the choice of the sequence and auxiliary problem (5.2). Theorem 5.1 shows that the Fredholm property of (7.3) follows from those of (5.3) and (7.2).

Conversely, if there is a space $H_T^{s-O-1/2}(\mathcal{Y})$ with the property that the operator (7.3) is Fredholm, then the operator $\Psi = B_{\mathcal{Y}} C_{\mathcal{Y},L}^{-1}$ is Fredholm in the spaces (7.2), and so generalised elliptic. We have thus arrived at

Theorem 7.2. *If an elliptic operator A possesses some boundary problem (5.2) satisfying the Shapiro-Lopatinskii condition, then the operator $B_{\mathcal{Y}}$ corresponding to the problem (5.1) for $f = 0$ is Fredholm in the spaces (7.3) if and only if this latter problem is generalised elliptic.*

As the range of the operator Ψ is contained in the range of the operator $B_{\mathcal{Y}}$, Theorem 5.1 readily implies

Corollary 7.3. *The null-space of operator (7.3) related to the boundary problem (5.1) with $f = 0$ consists of C^∞ functions in $\overline{\mathcal{X}}$, and its range is described by a condition of orthogonality to certain functions of class $C^\infty(\mathcal{Y})$.*

Moreover, if u is a weak solution of (5.1) with $f = 0$ and $B_{\mathcal{Y}} u \in H_T^{s-O-1/2}(\mathcal{Y})$, then $u \in H^{s+M}(\mathcal{X})$.

As usual, by the index $\operatorname{ind} B_{\mathcal{Y}}$ of operator (7.3) is meant the difference of the number of linearly independent solutions of the homogeneous problem (5.1) and the number of linearly independent vector-valued functions the orthogonality to which provides the solvability of the problem (5.1) with $f = 0$. By Theorem 7.2, $\operatorname{ind} B_{\mathcal{Y}}$ does not depend on s if $s > |O|$.

Theorem 7.4. *Suppose the problem (5.1) for $f = 0$ is generalised elliptic. Then the indices of operators (5.3), (7.2), and (7.3) satisfy*

$$\operatorname{ind} B_{\mathcal{Y}} = \operatorname{ind} \Psi + \operatorname{ind} C_{\mathcal{Y}}.$$

Proof. By the properties of index, we obtain from the equalities $\Psi = B_{\mathcal{Y}} C_{\mathcal{Y},L}^{-1}$ and (5.4) that

$$\begin{aligned} \operatorname{ind} \Psi &= \operatorname{ind} B_{\mathcal{Y}} + \operatorname{ind} C_{\mathcal{Y},L}^{-1}, \\ \operatorname{ind} C_{\mathcal{Y},L}^{-1} &= -\operatorname{ind} C_{\mathcal{Y}}, \end{aligned}$$

which establishes the formula. □

This is a simple generalisation of the well-known formula of [AD62] which compares the indices of two elliptic boundary value problems for the same differential operator in a domain.

8. Inhomogeneous problem

Our next objective is to study the inhomogeneous problem (5.1), i.e., the problem with arbitrary f. Following [VG67], we consider the auxiliary boundary problem

$$\left\{ \begin{array}{rcll} A(x,D)u(x) & = & f & \text{for} \quad x \in \mathcal{X}, \\ C(x,D)u(x) & = & 0 & \text{for} \quad x \in \mathcal{Y}. \end{array} \right. \tag{8.1}$$

Since it satisfies the Shapiro-Lopatinskii condition, the operator

$$A: \ H^{s+M}(\mathcal{X}) \cap \ker C_{\mathcal{Y}} \to H^{s-N}(\mathcal{X}) \tag{8.2}$$

is Fredholm. Hence, for each $f \in H^{s-N}(\mathcal{X})$ orthogonal to a finite number of smooth vector-valued functions $g_1, \ldots, g_{D'}$ on $\mathcal{X}$, there is a solution $u \in H^{s+M}(\mathcal{X})$ of (8.1). Denote by $f \mapsto Gf$ the operator which assigns, to any such f, the solution of (8.1) orthogonal to all solutions of the homogeneous problem (8.1). Obviously, such a solution Gf is unique. We extend G to an operator on all of $H^{s-N}(\mathcal{X})$ by linearity, setting $Gf = 0$ if f belongs to the cokernel of (8.2). Then G is a regulariser of the operator (8.2), i.e.,

$$\begin{array}{rclcl} GAu & = & u - H_0 u & \text{if} & Cu\,|_{\mathcal{Y}} = 0; \\ AGf & = & f - H_1 f & \text{and} & C\,Gf\,|_{\mathcal{Y}} = 0, \end{array} \tag{8.3}$$

where H_0 and H_1 are projections onto the kernel and cokernel of (8.2), respectively, both H_0 and H_1 being smoothing operators of order $-\infty$.

Assume that f is orthogonal to all $g_1, \ldots, g_{D'}$. We look for a solution of (5.1) of the form $u = Gf + U$. Then for U we obtain the problem (5.1) with $f = 0$,

$$\left\{ \begin{array}{rcll} A(x,D)U(x) & = & 0 & \text{for} \quad x \in \mathcal{X}, \\ B(x,D)U(x) & = & U_0(x) & \text{for} \quad x \in \mathcal{Y}, \end{array} \right. \tag{8.4}$$

where $U_0 = u_0 - B_{\mathcal{Y}} Gf \in H^{s-O-1/2}(\mathcal{Y})$, for $Gf \in H^{s+M}(\mathcal{X})$. Since (8.4) is a generalised elliptic problem, it has a solution U in $H^{s+M}(\mathcal{X})$ if $U_0 \in H_T^{s-O-1/2}(\mathcal{X})$ is orthogonal to a finite number of linearly independent vector-valued functions $v_{0,1}, \ldots, v_{0,D''}$ of class $C^\infty(\mathcal{Y})$, i.e.,

$$\int_{\mathcal{Y}} (v_{0,j}(x))^* U_0(x)\, ds = 0$$

for $j = 1, \ldots, D''$. Summarising we rewrite the conditions of solvability of problem (5.1) in the form

$$\begin{array}{rcll} \displaystyle\int_{\mathcal{X}} (g_j(x))^* f(x)\, dx & = & 0, & \text{for} \quad j = 1, \ldots, D'; \\ \displaystyle\int_{\mathcal{Y}} (v_{0,j}(x))^* (u_0(x) - B_{\mathcal{Y}} Gf(x))\, ds & = & 0, & \text{for} \quad j = 1, \ldots, D''. \end{array} \tag{8.5}$$

It is fairly complicated to show separate conditions on f and u_O which are necessary and sufficient for U_0 to belong to $H_T^{s-O-1/2}(\mathcal{Y})$. In other words, it is difficult to choose natural spaces in which the operator of a generalised elliptic problem (5.1) acts. Hence we restrict ourselves to an assertion which is not as explicit as Theorem 7.2.

Theorem 8.1. *If problem (5.1) is generalised elliptic of degree of non-ellipticity δ, then for each $s > \max\{o', p'\} - \delta + 1/2$ it has a solution in the space $H^{s+M}(\mathcal{X})$ whenever $f \in H^{s-N+n'}(\mathcal{X})$ and $u_0 \in H_T^{s-O-1/2}(\mathcal{Y})$ satisfy the orthogonality conditions (8.5).*

Here o' is the maximal order of normal differentiation in the operator B, p' is the maximal order of normal differentiation in the operator C, and n' is the maximal order of differentiation in the composition $T_n \dots T_1$ reducing $\Lambda^{-O}\Psi$ to an elliptic operator, $n' \leq |O| - \delta$.

The homogeneous problem corresponding to (5.1) has a finite number of linearly independent solutions, and these are C^∞ up to the boundary of $\mathcal{X}$. If u is a weak solution of (5.1), such that $Au \in H^{s-N+n'}(\mathcal{X})$ and $Bu\,|_{\mathcal{Y}} \in H_T^{s-O-1/2}(\mathcal{Y})$, then $u \in H^{s+M}(\mathcal{X})$.

Indeed, consider a weak solution u of (5.1) with these smoothness properties. By (8.3), the function $U := u - Gf$ is a solution of the problem

$$\left\{ \begin{array}{rcll} AU & = & F & \text{in } \mathcal{X}, \\ BU & = & U_0 & \text{on } \mathcal{Y}, \end{array} \right. \tag{8.6}$$

where $F = H_1 f \in C^\infty(\overline{\mathcal{X}}, \mathbb{C}^k)$ and $U_0 = u_0 - B_{\mathcal{Y}} Gf \in H_T^{s-O-1/2}(\mathcal{Y})$. Since Gf lies in $H^{s+M}(\mathcal{X})$, it suffices to show that $U \in H^{s+M}(\mathcal{X})$. To this end, we consider the regulariser Π of (5.3). From (5.4) and (8.6) we get

$$\begin{array}{rcll} A(U - \Pi C_{\mathcal{Y}} U) & = & H_1 f & \text{in } \mathcal{X}, \\ C_{\mathcal{Y}}(U - \Pi C_{\mathcal{Y}} U) & = & S_1 C_{\mathcal{Y}} U & \text{on } \mathcal{Y}, \end{array}$$

where $H_1 f$ and $S_1 C_{\mathcal{Y}} U$ are C^∞ functions on $\overline{\mathcal{X}}$ and $\mathcal{Y}$, respectively. By the smoothness of solutions of elliptic boundary problems we readily deduce that the difference $U - \Pi C_{\mathcal{Y}} U$ is C^∞ on the closure of $\mathcal{X}$. Applying the operator $B_{\mathcal{Y}}$ to this function and taking into account (8.6) we see that $\Psi C_{\mathcal{Y}} U = U_0$ up to a vector-valued function of class $C^\infty(\overline{\mathcal{X}})$. As Ψ is a generalised elliptic operator with principal symbol structure $(O, -P)$ and $U_0 \in H_T^{s-O-1/2}(\mathcal{Y})$, Corollary 7.3 yields $C_{\mathcal{Y}} U \in H^{s-P-1/2}(\mathcal{Y})$. Combining this with $AU \in C^\infty(\overline{\mathcal{X}}, \mathbb{C}^k)$ and once again using the regularity property of solutions of elliptic problems, we deduce $U \in H^{s+M}(\mathcal{X})$, which is the desired conclusion.

9. An example

Consider an inverse problem of Newton potential. When linearised, the problem of finding a domain and density by the pair of outer potentials p_1 and p_2 reduces to the following one: Given any positive harmonic functions p_1 and p_2 on an open set $U \subset \mathbb{R}^3$, find harmonic functions u_1 and u_2 in a domain $\mathcal{X} \subset\subset U$ which satisfy the boundary conditions

$$\begin{array}{rcl} p_1 u_1 - p_2 u_2 & = & u_D, \\ p_1 \dfrac{\partial u_1}{\partial \nu} - p_2 \dfrac{\partial u_2}{\partial \nu} + p_2 p u_2 & = & u_N \end{array} \tag{9.1}$$

on $\mathcal{Y}$, where ν is the unit inner normal vector to the boundary $\mathcal{Y}$, and

$$p = \frac{\partial}{\partial \nu} \log \frac{p_2}{p_1}.$$

We first show that (9.1) is a generalised elliptic problem provided that the function p does not vanish on $\mathcal{Y}$. The principal symbol structures (N, M) and (O, M) of operators $A = \Delta E_2$ and B given by (9.1) choose as follows: $N = (0,0)$, $M = (2,2)$, and $O = (-2,-1)$. Use the auxiliary Dirichlet problem. Then the operator Ψ reduces to an elliptic pseudodifferential operator of principal symbol structure $(O', -P) = ((-2,-2),(2,2))$. Choose orthogonal coordinates $x = (x_1, x_2, x_3)$ in a neighbourhood of a boundary point y_0, such that $\mathcal{X}$ be defined by the inequality $x_n = t \geq 0$, the lines parallel to the t-axis be geodesics orthogonal to $\mathcal{Y}$, and t would coincide with the arc length. As in (6.2), the Laplace operator splits as

$\Delta = \Delta_0 + \Delta_1 + \ldots$, where

$$\begin{aligned}
\Delta_0 &= \sum_{j=1}^{3} a_{j,j}(y_0)\,\frac{\partial^2}{\partial x_j^2},\\
\Delta_1 &= \sum_{j=1}^{3} \langle \nabla a_{j,j}(y_0), x-y_0\rangle \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^{3} a_j(y_0)\,\frac{\partial}{\partial x_j},
\end{aligned}$$

where $a_{j,j}$ and a_j are expressed through the coefficients of the metric tensor and $a_{3,3}=1$. In the same way for the matrix B of (9.1) we obtain $B = B_0 + B_1 + \ldots$, where

$$\begin{aligned}
B_0 &= \begin{pmatrix} p_1(y_0) & -p_2(y_0) \\ p_1(y_0)\dfrac{\partial}{\partial t} & -p_2(y_0)\dfrac{\partial}{\partial t} \end{pmatrix},\\
B_1 &= \begin{pmatrix} \langle \nabla p_1(y_0), x-y_0\rangle & -\langle \nabla p_2(y_0), x-y_0\rangle \\ \langle \nabla p_1(y_0), x-y_0\rangle \dfrac{\partial}{\partial t} & -\langle \nabla p_2(y_0), x-y_0\rangle \dfrac{\partial}{\partial t} + p_2(y_0)p(y_0) \end{pmatrix}.
\end{aligned}$$

The problem (6.3) for $q=0$ on the half-axis $t>0$

$$\begin{aligned}
\tilde{\Delta}_0 \varSigma_0 := \Big(|\eta|^2 - \frac{\partial^2}{\partial t^2}\Big)\varSigma_0(y_0,\eta;t) &= 0,\\
\varSigma_0(y_0,\eta;0) &= E_2
\end{aligned}$$

has a unique solution $\varSigma_0(y_0,\eta;t) = \exp(-|\eta|t)E_2$ which vanishes as $t\to+\infty$. For $q=1$ it has the form

$$\begin{aligned}
\tilde{\Delta}_0 \varSigma_1 &= -\tilde{\Delta}_1 \varSigma_0, \quad \text{for} \quad t>0,\\
\varSigma_1(y_0,\eta;0) &= 0,
\end{aligned}$$

where $\varSigma_1(y_0,\eta;t)\to 0$ as $t\to+\infty$. It is easy to compute that

$$\tilde{\Delta}_1 \varSigma_0 = (c_0 + c_1 t)\exp(-|\eta|t)E_2$$

whence $\varSigma_1(y_0,\eta;t) = t(C_0 + C_1 t)\exp(-|\eta|t)E_2$. The coefficients c_0, c_1 and C_0, C_1 are expressed through the coefficients of operators Δ_0 and Δ_1. However, the final result is actually independent of $\varSigma_1$, as we will see, and so we need not any explicit expressions.

By (6.4), the first two terms of the complete symbol of operator $\varPsi$ of order $(O,-P) = ((-2,-1),(2,2))$ has the form

$$\begin{aligned}
a_0 + a_1 &= \tilde{B}_0\varSigma_0\,(y_0,\eta;0) + \Big(\tilde{B}_1\varSigma_0\,(y_0,\eta;0) + \tilde{B}_0\varSigma_1\,(y_0,\eta;0)\Big)\\
&= \begin{pmatrix} p_1 & -p_2 \\ -p_1|\eta| & p_2|\eta| \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ p_1C_0 - \imath Q_1 & -(p_2C_0 - \imath Q_2) + p_2 p \end{pmatrix},
\end{aligned}$$

where

$$Q_i(y_0,\eta) = \sum_{j=1}^{2} \frac{\partial p_i}{\partial x_j}(y_0)\frac{\partial}{\partial \eta_j}|\eta|$$

for $i=1,2$. Note that both $C_0(y_0,\eta)$ and $Q_i(y_0,\eta)$ are of degree 0 in η.

Denote by $\varLambda^1$ a pseudodifferential operator with principal symbol $|\eta|$ and consider the matrix

$$T = \begin{pmatrix} 1 & 0 \\ \varLambda^1 & 1 \end{pmatrix}.$$

It is easy to check that $T\varPsi$ is a zero order pseudodifferential operator with principal symbol

$$\sigma^0(T\varPsi) = \begin{pmatrix} p_1 & -p_2 \\ p_1C_0 - \imath Q_1 & -(p_2C_0 - \imath Q_2) + p_2 p \end{pmatrix}.$$

Since

$$\det \sigma^0(T\varPsi) = p_1p_2p + \imath\,(p_1Q_2 - p_2Q_1)$$

on $\mathcal{Y}$ and the imaginary part $p_1 Q_2 - p_2 Q_1$, which is linear in η for $|\eta| = 1$, always vanishes for some η, the operator $T\Psi$ is elliptic if and only if p does not vanish on the boundary $\mathcal{Y}$.

Summarising, we conclude that Ψ is reducible to the operator $T\Psi$ of order 0, or principal symbol structure $(O', -P) = ((-2,-2),(2,2))$, as desired. Applying Theorem 7.2 yields

Theorem 9.1. *The operator $B_{\mathcal{Y}}$ of problem (9.1) is Fredholm in the spaces*

$$B_{\mathcal{Y}} : \, H^{s+(2,2)}(\mathcal{X}) \cap \ker(\Delta E_2) \to H_T^{s+(2,2)-1/2}(\mathcal{Y}),$$

for $s \geq 0$, if and only if $p\,|_{\mathcal{Y}} \neq 0$.

By $H_T^{s+(2,2)-1/2}(\mathcal{Y})$ is meant the space of generalised vector-valued functions $u_0 = (u_D, u_N)$, such that $Tu_0 \in H^{s+(2,2)-1/2}(\mathcal{Y})$. This just amounts to saying that both u_D and $\Lambda^1 u_D + u_N$ belong to $H^{s+3/2}(\mathcal{Y})$. Since the Dirichlet problem has index 0 and the operator T is invertible, we deduce from Theorem 7.4 that $\operatorname{ind} B_{\mathcal{Y}} = \operatorname{ind}(T\Psi)$.

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