SHAPIRO-LOPATINSKIJ CONDITION FOR ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract

Elliptic boundary value problems are well-posed in suitable Sobolev spaces, if the boundary conditions satisfy the Shapiro–Lopatinskij condition. We propose here a criterion, which covers also overdetermined elliptic systems, for checking this condition. We present a constructive method for computing the compatibility operator for the given boundary value problem operator which is also necessary when checking the criterion. In case of 2 independent variables we give a formulation of the criterion for the Shapiro–Lopatinskij condition which can be checked in finite number of steps.

Our approach is based on formal theory of PDEs, and we use constructive module theory and polynomial factorisation in our test. Actual computations were carried out with computer algebra systems Singular and MuPad.

1. *Introduction*

It is well-known that elliptic boundary (value) problems are well-posed only if the boundary conditions are chosen appropriately. By well-posedness one usually means that the solution exists and is unique in some space, and it depends continuously on data and parameters, or more generally that the relevant operator is at least Fredholm (kernel and cokernel are finite dimensional). The property which the boundary conditions should satisfy to have a well-posed problem in some Sobolev spaces for a elliptic boundary value problem is called the Shapiro–Lopatinskij condition. Of course in many physical models the boundary conditions are more or less clear, and if the model is at all reasonable one may expect that these "natural" boundary conditions give a well-posed problem. However, in more complicated models one may not have any natural boundary conditions, or it may not be clear which boundary conditions are "best" in a given situation.

In [**17**] Mohammadi and Tuomela proposed to use involutive form of the PDE system in numerical computations. The resulting systems are not standard ones, so the question naturally arose how to effectively check if given boundary conditions indeed satisfy the Shapiro–Lopatinskij condition. The purpose of the present article is to give a (partial) answer to that question.

Originally the Shapiro–Lopatinskij condition was formulated for square[1] elliptic systems [**16**], [**23**], but subsequently the theory was generalised to square DN–elliptic systems[2] [**2**] [**3**] [**4**] and overdetermined elliptic systems [**8**] [**9**]. Note that there is no need to consider overdetermined DN–elliptic systems because we, together with Seiler, proved in [**15**] that any DN–elliptic problem becomes elliptic when completed to involutive form.

The plan of the paper is as follows. In section 2 we review the necessary background, mainly some constructive module theory and formal theory of PDEs, but we also recall some facts about Sobolev spaces. In section 3 we prove some results related to the notion of finite type and introduce DN–elliptic systems. Then in section 4 we take up square DN–elliptic systems and propose a criterion for checking the Shapiro–Lopatinskij condition. We also give a constructive test for checking this condition in case of 2 independent variables. This restriction is due to the fact that in 2 variable case we can use factorisation of polynomials in *one* variable in a way which seems to be impossible in the general case. Note that strictly speaking we could have treated directly the general case, but because square DN–elliptic systems are anyway important in practice it is perhaps best to consider them separately. Also results of this section are helpful in proving more general results later. In section 5 we introduce boundary value problem operators, and show how one can constructively compute their compatibility operators. These constructions are based on computing syzygies using Gröbner bases and suitable

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[1] as many algebraically independent equations as unknowns
[2] elliptic in the sense of Douglis and Niremberg

module orderings. In section 6 we then give a new criterion for checking the Shapiro–Lopatinskij condition in the overdetermined case. We also propose a constructive test in 2 variable case. Finally in section 7 we draw some conclusions and indicate some directions for future work.

In our examples we have used Singular [**13**] and MuPad [**11**]. In the Appendix we show how to use these programs to perform the needed computations.

2. *Preliminaries*

2.1. *Algebra*

All the relevant material can be found for example in [**12**] and [**10**] (orderings and commutative algebra) and [**5**] and [**18**] (field extensions).

2.1.1. Orderings

Let $\mathbb{N}_0^n$ be the space of multi indices, i. e. the set of all ordered n-tuples $\mu = (\mu_1, \dots, \mu_n)$ with $\mu_i \in \mathbb{N}_0$. $\mathbf{1}_j$ is a multi index whose j'th component is one and others are zero. The *length* of a multi index is $|\mu| = \mu_1 + \cdots + \mu_n$ and the *class* of the multi-index μ, denoted by $\mathsf{cls}\,\mu$, is ℓ, if $\mu_1 = \cdots = \mu_{\ell-1} = 0$ and $\mu_\ell \neq 0$.

A total ordering $>$ on the set of monomials $\mathrm{Mon}_n = \{\xi^\mu \mid \mu \in \mathbb{N}_0^n\}$ in n variables is called a *global monomial ordering* if it satisfies the following conditions: (1) $\xi^\alpha > \xi^\beta$ implies $\xi^\gamma\xi^\alpha > \xi^\gamma\xi^\beta$ for all $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ and (2) $\xi^\alpha > 1$ for all $\alpha \neq 0$. In this paper we will only consider such orderings and for simplicity we will drop the words 'global monomial' from now on. We will need in the sequel the following orderings.

- *lexicographic ordering* (denoted by $>_{lp}$): $\xi^\alpha >_{lp} \xi^\beta$ if and only if
$$\exists\, 1 \leqslant i \leqslant n : \alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i;$$
- *degree reverse lexicographic ordering* (denoted by $>_{dp}$): $\xi^\alpha >_{dp} \xi^\beta$ if and only if
$$|\alpha| > |\beta| \text{ or } \Big(|\alpha| = |\beta| \text{ and } \exists\, 1 \leqslant i \leqslant n : \alpha_n = \beta_n, \dots, \alpha_{i+1} = \beta_{i+1}, \alpha_i < \beta_i\Big);$$
- *product ordering*: let $>_1$ be an ordering on $\mathrm{Mon}(\xi_1, \dots, \xi_n)$ and $>_2$ be an ordering on $\mathrm{Mon}(\eta_1, \dots, \eta_l)$. Then the product ordering $>$ on $\mathrm{Mon}(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_l)$ is defined as
$$\xi^\alpha\eta^\beta > \xi^{\alpha'}\eta^{\beta'} \quad \Leftrightarrow \quad \xi^\alpha >_1 \xi^{\alpha'} \text{ or } \Big(\alpha = \alpha' \text{ and } \eta^\beta >_2 \eta^{\beta'}\Big).$$

2.1.2. Commutative algebra

Let $\mathbb{A} = \mathbb{K}[\xi] = \mathbb{K}[\xi_1, \dots, \xi_n]$ be a polynomial ring in n variables where $\mathbb{K}$ is some field of characteristic zero. If $\mathcal{I}$ is an ideal of $\mathbb{K}[\xi]$, then $\mathbb{A}_\mathcal{I} = \mathbb{K}[\xi]/\mathcal{I}$ is the residue class ring.

The cartesian product $\mathbb{A}^k$ is an $\mathbb{A}$–module of rank k. A module which is isomorphic to some $\mathbb{A}^k$ is called *free*. A module M is *finitely generated*, if there are elements $a_1, \dots, a_\nu \in M$ such that $M = \langle a_1, \dots, a_\nu\rangle$. Every submodule of $\mathbb{A}^k$ is finitely generated.

Let C be a $m \times m$ matrix whose elements belong to $\mathbb{A}$. The *adjoint* of C, denoted by $\mathsf{adj}\,(C) \in \mathbb{A}^{m\times m}$, is the matrix of cofactors of C, i. e.
$$C\,\mathsf{adj}\,(C) = I_m \det(C)$$
where I_m is the unit matrix of size $m \times m$.

Let us consider a homomorphism $\varphi : \mathbb{A}^k \to \mathbb{A}^m$ and its image $M_0 = \mathrm{image}(\varphi) = \langle b^1, \dots, b^k\rangle \subset \mathbb{A}^m$. If $s \in \mathbb{A}^k$ is such that
$$s_1 b^1 + \cdots + s_k b^k = 0\ ,$$
then s is called a *syzygy* of M_0. All such s form the (first) *syzygy module* of M_0 which is denoted by M_1. But now M_1 is the image of some homomorphism φ_1, and one can consider its syzygies. Now Hilbert's syzygy theorem [**10**, p. 45] asserts that every finitely generated $\mathbb{A}$-module has a *free resolution*, i.e. an exact sequence of the form
$$0 \longrightarrow \mathbb{A}^{k_r} \longrightarrow \mathbb{A}^{k_{r-1}} \longrightarrow \cdots \xrightarrow{\varphi_2} \mathbb{A}^{k_1} \xrightarrow{\varphi_1} \mathbb{A}^k \xrightarrow{\varphi} \mathbb{A}^m \longrightarrow \mathbb{A}^m/M_0 \longrightarrow 0 \qquad (2.1)$$

Recall that exactness means that the image of any map in this sequence is equal to the kernel of the following map. The length of a free resolution is less than or equal to n, where n is the number of variables in the polynomial ring $\mathbb{A}$, i. e. for our module M_0 we have $r \leqslant n$.

2.1.3. Module orderings
Let $\mathbb{A} = \mathbb{K}[\xi]$ as before and let us denote by e^i the canonical basis vectors of $\mathbb{A}^k$. The elements of the form $\xi^\mu \mathsf{e}^i$ can be ordered in two different ways. Here $<$ can be any monomial ordering.

- *TOP ordering*

$$\xi^\alpha \mathsf{e}^i <_m \xi^\beta \mathsf{e}^j \quad \text{if} \quad \xi^\alpha < \xi^\beta \text{ or } (\alpha = \beta \text{ and } i > j).$$

- *POT ordering*

$$\xi^\alpha \mathsf{e}^i <_m \xi^\beta \mathsf{e}^j \quad \text{if} \quad i > j \text{ or } (i = j \text{ and } \xi^\alpha < \xi^\beta).$$

Now let $\hat{\mathbb{A}} = \mathbb{K}[\xi, \eta]$ and let $\hat{M}$ be some submodule of $\hat{\mathbb{A}}^k$. We choose a TOP module ordering in $\hat{\mathbb{A}}^k$ and product ordering in $\hat{\mathbb{A}}$ with η variables bigger than ξ. We will need the following fact [**1**, p. 156].

LEMMA 2.1. *If $\hat{G}$ is a Gröbner basis for $\hat{M}$, then $\hat{G} \cap \mathbb{A}^k$ is a Gröbner basis for $\hat{M} \cap \mathbb{A}^k$.*

Hence TOP orderings can be used to eliminate variables. On the other hand POT orderings can be used to eliminate components. Let M be a submodule of $\mathbb{A}^k = \mathbb{A}^i \oplus \mathbb{A}^{k-i}$. We choose a POT module ordering for $\mathbb{A}^k$, and any monomial ordering in $\mathbb{A}$. Then we have [**12**, p. 177]

LEMMA 2.2. *If G is a Gröbner basis for M, then $G \cap \mathbb{A}^{k-i}$ is a Gröbner basis for $M \cap \mathbb{A}^{k-i}$.*

2.1.4. Field extensions
An *extension* of a field $\mathbb{K}$ is a field $\mathbb{L}$ which contains $\mathbb{K}$ as a subfield. The extension $\mathbb{L}$ is called *algebraic* if every element of $\mathbb{L}$ is algebraic over $\mathbb{K}$, i. e. if every element of $\mathbb{L}$ is a root of some non-zero polynomial with coefficients in $\mathbb{K}$.

If $\alpha \in \mathbb{L} \backslash \mathbb{K}$ is algebraic over $\mathbb{K}$ then there is a unique monic polynomial p of least degree such that $p(\alpha) = 0$. This p is called a *minimal polynomial* of α.

The field extension $\mathbb{L}$ is called *finitely generated*, if there are elements $\alpha_1, \ldots, \alpha_n \in \mathbb{L} \setminus \mathbb{K}$ such that $\mathbb{L} = \mathbb{K}(\alpha_1, \ldots, \alpha_n)$. If a field extension $\mathbb{L}$ over $\mathbb{K}$ is generated by a single element α then α is called a *primitive* element.

The *splitting field* of a polynomial $p \in \mathbb{K}[x]$ is a field extension $\mathbb{L}$ of $\mathbb{K}$ over which p factorizes into linear factors $x - b_i$ and such that the b_i generate $\mathbb{L}$ over $\mathbb{K}$. Hence, the splitting field is finitely generated and moreover we have the *primitive element theorem* [**5**]:

THEOREM 2.1. *Let $\mathbb{K}$ be a field of characteristic 0. Then every finitely generated extension of $\mathbb{K}$ has a primitive element. In particular, the splitting field of any polynomial $p \in \mathbb{K}[x]$ has a primitive element.*

2.2. *Complex Analysis*
Let $\zeta_1, \ldots, \zeta_\nu \in \mathbb{C}$ be some points lying in the open upper half of the complex plane. Let us define following polynomials:

$$\begin{aligned} p^+ &= (\zeta - \zeta_1) \cdots (\zeta - \zeta_\nu) = \sum_{j=0}^{\nu} b_j \zeta^j \\ p_l^+ &= \sum_{j=l}^{\nu} b_j \zeta^{j-l}, \quad l = 1, \ldots, \nu, \end{aligned} \tag{2.2}$$

LEMMA 2.3 ([**2**]). *Let γ_+ be a simple closed curve oriented counterclockwise in the upper half of the complex plane surrounding all the roots of the polynomial p^+. Then*

$$\frac{1}{2\pi i} \oint_{\gamma_+} \frac{\zeta^\tau p_l^+(\zeta)}{p^+(\zeta)} d\zeta = \begin{cases} 1, & \tau = l - 1, \\ 0, & \tau \neq l - 1, \end{cases} \qquad \tau = 0, \ldots, \nu - 1,$$

Let $\mathfrak{M}_+$ be the space of functions $u : \mathbb{R} \to \mathbb{C}^m$ which tend to zero as $x_n \to +\infty$ and let v be a vector whose elements are in $\mathbb{C}[\zeta]$. Then we set

$$\omega^l(x_n) = \frac{1}{2\pi i} \oint_{\gamma_+} \frac{v(\zeta) p_l^+(\zeta) e^{i\zeta x_n}}{p^+(\zeta)}\, d\zeta, \quad l = 1, \ldots, \nu.$$

The following Lemma is similar to some results in [**14**]. However, since we did not find anywhere the precise result that we needed, we give the proof below.

LEMMA 2.4. *Let us suppose that v is not zero modulo p^+. Then $\omega^l \in \mathfrak{M}_+$, $l = 1, \ldots, \nu$, are linearly independent.*

Proof. First note that $\omega^l \in \mathfrak{M}_+$, since $\Re(i\zeta) < 0$. Let us now consider a linear combination

$$c_1\omega^1 + \cdots + c_\nu \omega^\nu = 0. \tag{2.3}$$

Since v is not divisible by p^+, we have

$$v = q\, p^+ + \mathsf{v}\,, \quad \mathsf{v} = \sum_{\tau=0}^{\nu-1} \mathsf{v}^\tau \zeta^\tau\,, \quad \mathsf{v} \neq 0,$$

where q, v and v^τ are some vectors. Let τ_0, $0 \leqslant \tau_0 < \nu$, be such that

$$\mathsf{v}^{\tau_0} \neq 0, \quad \mathsf{v}^\tau = 0 \text{ for all } 0 \leqslant \tau < \tau_0. \tag{2.4}$$

So we get

$$\omega^l(x_n) = \sum_{\tau=0}^{\nu-1} \mathsf{v}^\tau \frac{1}{2\pi i} \oint_{\gamma_+} \frac{\zeta^\tau p_l^+(\zeta) e^{i\zeta x_n}}{p^+(\zeta)}\, d\zeta, \quad l = 1, \ldots, \nu.$$

Lemma 2.3 implies that

$$\frac{d^k\omega^l}{dx_n^k}(0) = \begin{cases} i^k\, \mathsf{v}^{l-k-1} & ,\ l > k, \\ 0 & ,\ l \leqslant k, \end{cases} \tag{2.5}$$

Differentiating expression (2.3) $\nu - \tau_0 - 1$ times, substituting $x_n = 0$ and using (2.5), we get

$$\sum_{l=\nu-\tau_0}^{\nu} c_l \mathsf{v}^{l-\nu+\tau_0} = 0.$$

Applying (2.4), we obtain $c_\nu = 0$. Now differentiating expression (2.3) $\nu - \tau_0 - 2$ times, substituting $x_n = 0$ and using (2.5) and the fact that $c_\nu = 0$, we get

$$\sum_{l=\nu-\tau_0-1}^{\nu-1} c_l \mathsf{v}^{l-\nu+\tau_0+1} = 0.$$

But now (2.4) implies that $c_{\nu-1} = 0$. Continuing in this way we get $c_\tau = 0$ for $\tau_0 < \tau \leqslant \nu$. So (2.3) has now the following form

$$c_1\omega^1 + \cdots + c_{\tau_0}\omega^{\tau_0} = 0. \tag{2.6}$$

Now Lemma 2.3 implies also

$$\underbrace{\int_{x_n}^{+\infty} \cdots \int_{x_n}^{+\infty}}_{k} \omega^l(x_n) dx_n|_{x_n=0} = \begin{cases} i^{-k}\, \mathsf{v}^{l+k-1} & ,\ l \leqslant \nu - k, \\ 0 & ,\ l > \nu - k. \end{cases} \tag{2.7}$$

Integrating (2.6) once, substituting $x_n = 0$ and using (2.7), we get

$$\sum_{l=1}^{\tau_0} c_l \mathsf{v}^l = 0.$$

Hence, $c_{\tau_0} = 0$. Continuing in this way we finally get $c_i = 0$ for $1 \leqslant i \leqslant \tau_0$. □

2.3. *Geometry*

Here we simply give the basic definitions and refer to [**21**] and [**19**] for more details. All maps and manifolds will be assumed smooth, i.e. infinitely differentiable.

Let $\pi : \mathcal{E} \to \Omega$ be a bundle where $\mathcal{E}$ is called the *total space*, Ω the *base space* and π the *projection*. For each $p \in \Omega$, the set $\mathcal{E}_p = \pi^{-1}(p)$ is called the *fiber* over p. All fibers are diffeomorphic to each other. A *vector bundle* is bundle whose fibers are vector spaces. From now on we will assume that all bundles are vector bundles. For example we will need the following vector bundles: the tangent bundle $T\Omega$, the cotangent bundle $T^*\Omega$, and the bundle of symmetric q-forms $S^q(T^*\Omega)$.

Given two bundles $\pi_0 : \mathcal{E}_0 \to \Omega_0$ and $\pi_1 : \mathcal{E}_1 \to \Omega_1$ the map $\Phi : \mathcal{E}_0 \to \mathcal{E}_1$ is a *bundle map* or *morphism* if there is a map φ such that the following diagram commutes[3]

$$\begin{array}{ccc} \mathcal{E}_0 & \xrightarrow{\Phi} & \mathcal{E}_1 \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ \Omega_0 & \xrightarrow{\varphi} & \Omega_1 \end{array}$$

and the map $\mathcal{E}_0|_x \to \mathcal{E}_1|_{\varphi(x)}$ is linear. In many cases of interest we have $\Omega_0 = \Omega_1$ and $\varphi = \mathsf{id}$.

Let $\pi^q : J_q(\mathcal{E}) \to \Omega$ be the bundle of q-jets of $\pi : \mathcal{E} \to \Omega$. If the coordinates of the base space are denoted by $(x_1, \dots, x_n)$ and the coordinates of the fiber by $(y^1, \dots, y^m)$, then the coordinates of $J_q(\mathcal{E})$ (called jet coordinates) are denoted by

$$(x_1, \dots, x_n, y^1, \dots, y^m, \dots, y^\ell_\mu, \dots)$$

where $|\mu| \leqslant q$. Then denoting by n_q the number of distinct multi indices of length $|\mu| = q$, the dimension of $J_q(\mathcal{E})$ is $n + md_q$ where

$$d_q = 1 + n_1 + \dots + n_q = \binom{n+q}{q} \qquad \text{and} \qquad n_q = \binom{n+q-1}{q} \ .$$

Let us also introduce the standard projections

$$\pi^{q+r}_q : J_{q+r}(\mathcal{E}) \to J_q(\mathcal{E})$$

and define the embedding ε_q by requiring that the following complex be exact:

$$0 \longrightarrow S^q(T^*\Omega) \otimes \mathcal{E} \xrightarrow{\varepsilon_q} J_q(\mathcal{E}) \xrightarrow{\pi^q_{q-1}} J_{q-1}(\mathcal{E}) \longrightarrow 0$$

Recall that a *complex* $\mathcal{C}$ is a sequence of bundles $\mathcal{E}_i$ and bundle maps Φ_i such that

$$\mathcal{C} \quad : \quad 0 \longrightarrow \mathcal{E}_0 \xrightarrow{\Phi_0} \mathcal{E}_1 \xrightarrow{\Phi_1} \mathcal{E}_2 \xrightarrow{\Phi_2} \cdots$$

and $\Phi_{i+1}\Phi_i = 0$. Complex is *exact*, if $\mathrm{im}(\Phi_i) = \ker(\Phi_{i+1})$ for all i.

Finally a *section* of the bundle $\pi : \mathcal{E} \to \Omega$ is a map $f : \Omega \to \mathcal{E}$ such that $\pi \circ f = \mathsf{id}$. If f is a section of $\mathcal{E}$ then its q'th *prolongation*, a section of $J_q(\mathcal{E})$, is denoted by $j^q f$. The vector space of smooth sections of $\mathcal{E}$ is denoted by $C^\infty(\mathcal{E})$ and the Sobolev spaces of sections of $\mathcal{E}$ by $H_\alpha(\mathcal{E})$. However, it is essential to consider also the case where different components of the sections are in different Sobolev spaces. Hence we consider α as a vector:

$$y \in H_\alpha(\mathcal{E}) \quad \Longleftrightarrow \quad y^i \in H_{\alpha_i}(\mathcal{E}^i)$$

where we have the direct sum decomposition $\mathcal{E} = \oplus_i \mathcal{E}^i$. The corresponding norms are then defined by

$$||y||_\alpha = \sum_i ||y^i||_{\alpha_i}.$$

If we do not want to specify the space of sections or if this choice is irrelevant we will use the notation $\mathsf{S}(\mathcal{E})$.

[3] Strictly speaking the pair (Φ, φ) is the bundle map. However, φ is uniquely determined by Φ (if it exists).

2.4. *PDE*

All relevant materials can be found in [**9**], [**19**] and [**25**].

DEFINITION 2.1. *A (partial) differential system (or equation) of order q on $\mathcal{E}$ is a subbundle $\mathcal{R}_q$ of $J_q(\mathcal{E})$. Solutions of $\mathcal{R}_q$ are its (local) sections.*

We will only consider linear problems in the present article, so $\mathcal{R}_q$ will be a vector bundle.

Above we defined what the differential equations are, but we have not yet introduced any equations. To this end let us introduce two (vector) bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ and consider the following bundle map

$$\mathcal{A} \,:\, J_q(\mathcal{E}_0) \to \mathcal{E}_1$$

Note that $\mathcal{A}$ is a map between finite dimensional spaces. Then we can define a linear q'th order differential operator by the formula $A = \mathcal{A}\, j^q$:

$$A \,:\, \mathsf{S}(\mathcal{E}_0) \to \mathsf{S}(\mathcal{E}_1). \tag{2.8}$$

Now with $\mathcal{A}$ one can represent a differential equation as a zero set of a bundle map:

$$\mathcal{R}_q = \ker(\mathcal{A}) \quad : \quad \mathcal{A}(x, y, \dots) = 0 \tag{2.9}$$

DEFINITION 2.2. *The differential operator $j^r A \,:\, \mathsf{S}(\mathcal{E}_0) \to \mathsf{S}\big(J_r(\mathcal{E}_1)\big)$ is* the rth prolongation of A. *The associated morphism is denoted by $\mathcal{A}_r$.*

Then we can define the *prolongations* of $\mathcal{R}_q$ by

$$\mathcal{R}_{q+r} = \ker(\mathcal{A}_r)$$

Also we define

$$\mathcal{R}^{(s)}_{q+r} = \pi^{q+r+s}_{q+r}\big(\mathcal{R}_{q+r+s}\big)$$

Note that $\mathcal{R}^{(s)}_{q+r} \subset \mathcal{R}_{q+r}$, but in general these sets are not equal.

DEFINITION 2.3. *A differential operator A is called* sufficiently regular *if $\mathcal{R}^{(s)}_{q+r}$ is a vector bundle for all $r \geqslant 0$ and $s \geqslant 0$.*

If $\Omega \subset \mathbb{R}^n$ and the operator A has constant coefficients, then A is sufficiently regular.

DEFINITION 2.4. *A differential operator A (of order q) is* formally integrable *if A is sufficiently regular and $\mathcal{R}^{(1)}_{q+r} = \mathcal{R}_{q+r}$ for all $r \geqslant 0$.*

The formal integrability of an operator A of order q means that for any $r \geqslant 1$, all the differential consequences of order $q + r$ of the relations $Ay = 0$ may be obtained by way of differentiations of order no more than r and application of linear algebra.

Now it is well-known that some properties of PDEs depend only on the highest order derivative terms in the system. The information of this highest order part is coded in the symbol of the system. In fact there are two different kinds of symbols which are of interest here: the geometric symbol and the principal symbol.

DEFINITION 2.5. *Let us consider a sufficiently regular differential equation $\mathcal{R}_q \subset J_q(\mathcal{E})$ given by $\mathcal{R}_q = \ker(\mathcal{A})$.*

- *The* (geometric) symbol $\mathcal{M}_q$ *of $\mathcal{R}_q$ is a subbundle of $S^q(T^*\Omega)$ which is defined by the commutative and exact diagram* (2.10)*.*
- *The* principal symbol A *of operator* (2.8) *is a map*

$$\mathsf{A} = \mathcal{A}\,\varepsilon_q : S^q(T^*\Omega) \otimes \mathcal{E}_0 \to \mathcal{E}_1$$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{M}_q & \longrightarrow & \mathcal{R}_q & \xrightarrow{\pi^q_{q-1}} & J_{q-1}(\mathcal{E}_0) & & \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & S^q(T^*\Omega)\otimes\mathcal{E}_0 & \xrightarrow{\varepsilon_q} & J_q(\mathcal{E}_0) & \xrightarrow{\pi^q_{q-1}} & J_{q-1}(\mathcal{E}_0) & \longrightarrow & 0 \\
 & & \downarrow{\mathsf{A}} & & \downarrow{\mathcal{A}} & & & & \\
 & & \mathcal{E}_1 & = & \mathcal{E}_1 & & & &
\end{array}
\tag{2.10}
$$

In [**19**], [**22**] and [**24**] one can find more information about symbols.

We will now describe the symbol in a given coordinate system. Consider a linear q'th order PDE given by

$$
\mathcal{R}_q \quad : \quad Ay = \sum_{|\mu|\leqslant q} a_\mu(x)\partial^\mu y = f \tag{2.11}
$$

where $x \in \Omega \subset \mathbb{R}^n$, Ω is open, $a_\mu(x) \in \mathbb{R}^{k\times m}$ and $\mu \in \mathbb{N}_0^n$. We will always suppose that $k \geqslant m$. Now let M_q be the following matrix

$$
M_q = \left(a_{\mu^1}, a_{\mu^2}, \dots, a_{\mu^{n_q}}\right)
$$

where $\mu^1 > \mu^2 > \dots > \mu^{n_q}$ and $|\mu^i| = q$. In this way the (geometric) symbol $\mathcal{M}_q$ is defined by the kernel of M_q and we may also call the matrix M_q the symbol of $\mathcal{R}_q$. However, usually one considers A in a different way. Fixing some one form ξ we get a bundle map $\mathsf{A}(\xi) : \mathcal{E}_0 \to \mathcal{E}_1$ which in coordinates is given by

$$
\mathsf{A}(x,\xi) = \sum_{|\mu|=q} a_\mu(x)\xi^\mu
$$

REMARK 2.1. *We can interprete ξ in 4 different ways: (1) as a one form, i.e. a section of $T^*\Omega$; (2) the value of this one form at a given point p: $\xi \in T^*_p\Omega$; (3) a vector in $\mathbb{R}^n$, i.e. a coordinate representation of (2); and finally (4) indeterminates in some polynomial ring. We think that the intended interpretation will be clear from the context.*

In terms of matrices the connection between the two symbols is given by the formula

$$
\mathsf{A} = M_q(\Xi^q \otimes I) \tag{2.12}
$$

where $\Xi^q = \left(\xi^{\mu^1}, \xi^{\mu^2}, \dots, \xi^{\mu^{n_q}}\right)$. Hence algebraically we consider A as a module M_{A} generated by its rows: we may write this as $M_{\mathsf{A}} \subset \mathbb{A}^m$. All computations with principal symbol are based on this interpretation.

Anyway, it turns out that the symbol contains information which helps us to recognize if the given system is formally integrable. The relevant property is called the *involutivity* of the symbol. However, the actual definition of involutivity is a bit complicated, and because we will not need it explicitly we just refer to [**24**], [**19**], [**22**] and [**8**] for the actual definition.

The next result shows why the involutivity of the symbol is important.

THEOREM 2.2. *Let us suppose that the symbol $\mathcal{M}_q$ is involutive and that $\mathcal{R}_q$ is sufficiently regular. If no new integrability conditions are obtained by differentiating the system $\mathcal{R}_q$ once, then it is formally integrable.*

Now we say that the system $\mathcal{R}_q$ is *involutive* if it is formally integrable and the symbol $\mathcal{M}_q$ is involutive.

The above discussion suggests the following algorithm to compute the involutive form of a given system:

1. The system is prolonged until its symbol becomes involutive.
2. The system is prolonged and projected once to check if there are integrability conditions.
3. If there are no new equations in the previous step, the system is now involutive. Otherwise go back to step one with the system obtained in step two.

This is often called the *Cartan–Kuranishi completion algorithm*. One can show that under appropriate hypothesis the above algorithm terminates, in other words

THEOREM 2.3. *For a given sufficiently regular system $\mathcal{R}_q$ there are numbers r and s such that $\mathcal{R}_{q+r}^{(s)}$ is involutive.*

In practise to complete a system to the involutive form we have used the DETools package [**6**] in MuPAD [**11**] (see also Appendix 8.6 for appropriate MuPad commands).

2.5. *Functional analysis*

We have now defined differential equations and operators, but have not discussed any boundary conditions yet. To consider boundary value problems we introduce bundles $\mathcal{E}_i \to \Omega$ where Ω is now a manifold with boundary; we denote the boundary by Γ. Further let $\mathcal{G}_i \to \Gamma$ be some bundles on the boundary. The bundle $\mathcal{E}_i|_\Gamma \to \Gamma$ is the restriction of $\mathcal{E}_i \to \Omega$ to the boundary. If f is a section of $\mathcal{E}_i \to \Omega$, then γf is the corresponding section of $\mathcal{E}_i|_\Gamma \to \Gamma$; γ is called the *trace map*.

DEFINITION 2.6. *An operator of the form*

$$\Phi\,:\,\mathsf{S}(\mathcal{E}_0)\times\mathsf{S}(\mathcal{G}_0)\to\mathsf{S}(\mathcal{E}_1)\times\mathsf{S}(\mathcal{G}_1)\quad,\quad \Phi(y,z)=(\Phi^{11}y,\gamma\Phi^{21}y+\Phi^{22}z) \tag{2.13}$$

where Φ^{ij} are differential operators, is called a differential boundary operator *(DB-operator). If $\mathcal{G}_0=0$, we will write*

$$\Phi(y)=(Ay,By) \tag{2.14}$$

In this case Φ is called a boundary value problem operator *(BV-operator).*

REMARK 2.2. *Strictly speaking we should write γB instead of B. However, this is the standard way of writing the BV–operator.*

In terms of Sobolev spaces the most general DB–operator can be written as follows:

$$\Phi\,:\,H_\alpha(\mathcal{E}_0)\times H_\beta(\mathcal{G}_0)\to H_\delta(\mathcal{E}_1)\times H_\eta(\mathcal{G}_1) \tag{2.15}$$

where α, β, δ and η are some appropriately chosen vectors.

3. *Ellipticity of the symbol*

Let us consider the system (2.11). In the geometric definition of differential equations it is natural to assume that for each $x\in\Omega\cup\Gamma\subset\mathbb{R}^n$ the elements of the matrices $a_\mu(x)$ are real. However, to formulate our problem algebraically we need to assume that the elements of a_μ are in some field. Since our computations are local in nature it is best to think that we fix some x and then choose some convenient field. Obviously this field should be the same for all x. Now for purposes of symbolic computations we cannot take the field of real numbers; instead we will work with the field $\mathbb{K}$ which is of characteristic zero and a subfield of real numbers. So from now on we suppose that $a_\mu\in\mathbb{K}^{k\times m}$ and $k\geqslant m$.

Note that although the field $\mathbb{K}$ will be the same for all x, various objects that we will define, like characteristic polynomial, will a priori depend on x. However, for simplicity of notation this dependence on x will be suppressed.

Let $\mathsf{A}_1,\dots,\mathsf{A}_r$ be all $m\times m$ submatrices of A and let us denote the corresponding minors by $p_i=\det(\mathsf{A}_i)\in\mathbb{K}[\xi]$. Further let us define the (Fitting) ideal generated by these minors:

$$\mathcal{I}_{\mathsf{A}}=\langle p_1,\dots,p_r\rangle$$

The complex projective variety $V_{\mathbb{CP}}(\mathcal{I}_{\mathsf{A}})$ is called the *characteristic variety* of the operator A.

DEFINITION 3.1. *The differential operator A is called* elliptic *in Ω, if A is injective for all real $\xi\neq 0$ and for all $x\in\Omega$.*

Geometrically this condition means that the real projective variety $V_{\mathbb{RP}}(\mathcal{I}_{\mathsf{A}})=\emptyset$.

DEFINITION 3.2. *The* characteristic polynomial p_{A} *of an elliptic operator A is the greatest common divisor of its minors:*

$$p_{\mathsf{A}} = \mathsf{gcd}(p_1, \dots, p_r). \tag{3.1}$$

The greatest common divisor can be computed using Gröbner bases [**1**, p. 71]. It is immediate that $\mathcal{I}_{\mathsf{A}} \subset \langle p_{\mathsf{A}} \rangle$ and it is also easy to see that

LEMMA 3.1. *If $n > 1$, then the characteristic polynomial does not change when the symbol is prolonged.*

Proof. If A is the symbol, then $\Xi^r \otimes \mathsf{A}$ is the r–times prolonged symbol. From this the claim follows. □

It is well known that $\mathcal{I}_{\mathsf{A}}$ depends only on the module generated by the rows of A. The same property holds also for the characteristic polynomial.

THEOREM 3.1. *Characteristic polynomial p_{A} depends only on the module generated by the rows of* A.

Proof. Let us denote the rows of A by a^i, and let $M = \langle a^1, \dots, a^k \rangle$ be the module generated by these rows. Let $c \in M$ and A_c be A with the added row c. Obviously p_{A_c} divides p_{A}. We will show that also p_{A} divides p_{A_c}.

Let us denote the minors of A_c by q_i, if they are also minors of A and by $\tilde{q}_i$ otherwise. We need to show that p_{A} divides all $\tilde{q}_i$. Now $c = \sum_{i=1}^k p_i a^i$ where p_i are some polynomials. Without loss of generality we can write any $\tilde{q}_i$ as

$$\tilde{q}_i = \det\big(\sum_{j=1}^k p_j a^j, c^1, \dots, c^{m-1}\big)$$

where c^i's are some rows of A. But then

$$\tilde{q}_i = \sum_{j=1}^k p_j \det\big(a^j, c^1, \dots, c^{m-1}\big) = \sum_{j=1}^k p_j q_j$$

Hence $p_{\mathsf{A}} = p_{\mathsf{A}_c}$.

Now suppose that we have another basis of the module: $M = \langle b^1, \dots, b^s \rangle$, and let us denote the corresponding matrix by B. But all b^i's can be represented as $b^i = \sum_{j=1}^k p_{ij} a^j$. But by the previous argument adding such sums to the matrix does not change the characteristic polynomial. But then interchanging the roles of A and B we see that $p_{\mathsf{A}} = p_{\mathsf{B}}$. □

3.1. *Finite type*

DEFINITION 3.3. *Symbol $\mathcal{M}_q$ is of finite type, if $\mathcal{M}_{q+r} = 0$ for some r.*

Recall that if $\mathcal{M}_q$ is involutive and of finite type, then $\mathcal{M}_q = 0$. We have the following characterisation.

THEOREM 3.2. *The symbol is of finite type if and only if* $\mathsf{rad}(\mathcal{I}_{\mathsf{A}}) = \langle \xi_1, \dots, \xi_n \rangle$.

Geometrically the condition $\mathsf{rad}(\mathcal{I}_{\mathsf{A}}) = \langle \xi_1, \dots, \xi_n \rangle$ means that the characteristic variety is empty (as a projective variety).

Proof. First suppose that $\mathsf{rad}(\mathcal{I}_{\mathsf{A}}) \neq \langle \xi_1, \dots, \xi_n \rangle$. Hence there are $\xi \neq 0$ and $v \neq 0$ such that $\mathsf{A}(\xi)v = 0$. But then by formula (2.12) $M_q(\Xi^q \otimes v) = 0$. But then evidently $M_{q+r}(\Xi^{q+r} \otimes v) = 0$ for all r, so $\mathcal{M}_q$ is not of finite type.

Then suppose that $\mathsf{rad}(\mathcal{I}_{\mathsf{A}}) = \langle \xi_1, \dots, \xi_n \rangle$. Let us denote by e^i the canonical basis vectors of $\mathbb{A}^m$, and by $M_{\mathsf{A}} \subset \mathbb{A}^m$ the module generated by the rows of A. Now operating with adjoints of submatrices of A it is seen that $p_i \mathsf{e}^j \in M_{\mathsf{A}}$ for all i and j. This implies that if $f \in \mathcal{I}_{\mathsf{A}}$, then $f\,\mathsf{e}^j \in M_{\mathsf{A}}$ for all j. Hence by hypothesis there is s such that $\xi_i^{q+s} \mathsf{e}^j \in M_{\mathsf{A}}$ for all i and j. But then there must be some r such that $\xi_i^{\mu} \mathsf{e}^j \in M_{\mathsf{A}}$ for all $|\mu| = q + r$ and j. Hence the symbol $\mathcal{M}_{q+r}$ has a representation such that the symbol matrix M_{q+r} contains an identity matrix with $n_{q+r}m$ columns. Hence $\ker(M_{q+r}) = 0$. □

COROLLARY 3.1. *If $n > 1$ and $\mathcal{M}_q$ is of finite type, then $p_{\mathsf{A}} = 1$.*

Proof. Since $\mathcal{I}_{\mathsf{A}} \subset \langle p_{\mathsf{A}} \rangle$ we have $V_{\mathbb{CP}}(\langle p_{\mathsf{A}} \rangle) \subset V_{\mathbb{CP}}(\mathcal{I}_{\mathsf{A}})$. But if $p_{\mathsf{A}} \neq 1$, then $V_{\mathbb{CP}}(\langle p_{\mathsf{A}} \rangle)$ is not empty. □

In general the converse of the above result is not true. For example if $p_1 = \xi_1^2+\xi_2^2+\xi_3^2$ and $p_2 = 2\xi_1^2+\xi_2^2+\xi_3^2$, then

$$\mathsf{gcd}(p_1,p_2) = 1 \quad \text{but} \quad \mathsf{rad}(\langle p_1,p_2 \rangle) = \langle \xi_1, \xi_2^2+\xi_3^2 \rangle$$

However, we have the following partial converse.

LEMMA 3.2. *In the 2 dimensional case $p_{\mathsf{A}} = 1$ implies that $\mathcal{M}_q$ is of finite type.*

Proof. Let $f_i(\xi_1) = p_i(\xi_1, 1)$ and $g_i(\xi_2) = p_i(1, \xi_2)$. Now if $p_{\mathsf{A}} = 1$ we have also

$$\mathsf{gcd}(f_1,\dots,f_r) = 1 \quad \text{and} \quad \mathsf{gcd}(g_1,\dots,g_r) = 1$$

But this implies that there are polynomials $a_i \in \mathbb{K}[\xi_1]$ and $b_i \in \mathbb{K}[\xi_2]$ such that

$$\sum_{i=1}^{r} a_i f_i = 1 \quad \text{and} \quad \sum_{i=1}^{r} b_i g_i = 1$$

Homogenizing these equations show that these are numbers s_i such that $\xi_i^{s_i} \in \mathcal{I}_{\mathsf{A}}$. Hence $\mathsf{rad}(\mathcal{I}_{\mathsf{A}}) = \langle \xi_1, \xi_2 \rangle$. □

One may interprete the above results geometrically as follows. In a typical situation our problem is given in some domain $\Omega \subset \mathbb{R}^n$, and the boundary Γ is a $n-1$ dimensional submanifold of the closure of Ω. Hence the codimension of Γ is one. Now the fact that $p_{\mathsf{A}} = 1$ implies that one cannot impose any conditions in a set $S \subset \Omega \cup \Gamma$ of codimension one. However, it may be possible to give some conditions, if codim$(S) > 1$. But if $n = 2$ and codim$(S) > 1$ then dim$(S) = 0$, i.e. the system is of finite type. On the other hand if $n > 2$ the "intermediate" situations codim$(S) > 1$ and dim$(S) > 0$ can really occur. In [**4**] and [**20**] one can find some results for these kind of problems.

3.2. *Weights*

To generalise the notion of ellipticity, Douglis and Nirenberg [**7**] introduced the concept of *weights* of the system. In [**15**] it is shown that any system that is elliptic with respect to the generalised definition becomes elliptic when completed to the involutive form. So the apparent generality of ellipticity is just the result of restricting the attention to square systems. Hence this concept is interesting only in the square case, so when discussing DN–elliptic systems (elliptic in the sense of Douglis–Nirenberg) we will always suppose that $k = m$.

The weights are two sets of integers: we denote by s_l the weights for the equations, $1 \leqslant l \leqslant m$, and t_j the weights for the unknowns, $1 \leqslant j \leqslant m$. They must be chosen such that

$$s_l + t_j \geqslant q_{lj},$$

where q_{lj} is the maximal order of a derivative of the jth unknown function in the lth equation and if $s_l + t_j < 0$ then there are no derivatives of the jth unknown function in the lth equation of the system.

DEFINITION 3.4. *The* weighted (principal) symbol *of the differential operator A is*

$$\left(\mathsf{A}_w\right)_{l,j} = \sum_{|\mu| = s_l + t_j} \left(a_\mu(x)\right)_{l,j} \xi^\mu .$$

A is DN–elliptic, if A_w[4] *is injective for all real $\xi \neq 0$ and for all $x \in \Omega$.*

In particular, ordinary ellipticity is a special case of DN–ellipticity with weights

$$s_1 = \dots = s_m = 0 \quad \text{and} \quad t_1 = \dots = t_m = q .$$

Note that without loss of generality one may suppose, if convenient, that $s_1 \leqslant s_2 \leqslant \dots \leqslant s_m = 0$ and $t_1 \geqslant t_2 \geqslant \dots \geqslant t_m \geqslant 1$.

It is not difficult to see that DN–ellipticity is not invariant with respect to changing the dependent variables.

The following is a natural analog of Definition 3.2.

[4]Note that in all cases we mean the submatrix of A except the notation A_w which we use for the weighted principal symbol

DEFINITION 3.5. *The* characteristic polynomial *of a DN-elliptic operator A is*

$$p_{\mathsf{A}} = \det(\mathsf{A}_w).$$

This is a homogeneous polynomial in $\mathbb{K}[\xi]$ of degree $\sum_{l=1}^{m}(s_l + t_l)$.

Now whether the characteristic polynomial is defined by Definition 3.2 or by 3.5 we will need the following property. Let us set $\xi = (\xi', \zeta)$ so when we fix some ξ' we may view p_{A} as a polynomial of single variable ζ. Since $\xi' \in \mathbb{R}^{n-1}$, $p_{\mathsf{A}} \in \mathbb{R}[\zeta]$ in this interpretation, and p_{A} will be different for each different ξ'. However, we will still use the symbol p_{A} and hope that the intended meaning will be clear from the context. Similar convention will be used later on with other objects.

DEFINITION 3.6. *The operator A is* properly elliptic *(resp.* properly DN-elliptic*), if it is elliptic (resp. DN-elliptic) and at all boundary points and for all $\xi' \neq 0$ its characteristic polynomial $p_{\mathsf{A}} \in \mathbb{R}[\zeta]$ has equally many roots in the upper and lower half of the complex plane.*

Hence, for properly (DN-)elliptic operator A, the degree of characteristic polynomial p_{A} must be even. From now on we will always suppose that $p_{\mathsf{A}} \neq 1$ and denote the degree of charactristic polynomial by $2\nu > 0$.

Note that for $n > 2$, proper (DN-)ellipticity follows from (DN-)ellipticity. If the coefficients of the characteristic polynomial are real functions, then proper ellipticity follows from ellipticity also in case $n = 2$. In this paper we will only consider systems with real coefficients so for our purposes ellipticity and proper ellipticity are equivalent. Hence we will drop the word "proper" from now on.

Consider now a boundary operator

$$By = g, \quad x \in \Gamma, \tag{3.2}$$

where B is of size $\tilde{\nu} \times m$. For the purposes of this paper we may suppose that Γ is smooth. To define the weighted principal symbol of the boundary operator we introduce a third set of integers $r_1, \ldots, r_\nu$ such that

$$r_l + t_j \geqslant q^b_{lj},$$

where q^b_{lj} is the maximal order of a derivative of the jth unknown function in the lth boundary condition and if $r_l + t_j < 0$ then there are no derivatives of the jth unknown function in the lth boundary condition.

Then the *weighted (principal) symbol* of the boundary operator B is

$$\left(\mathsf{B}_w\right)_{l,j} = \sum_{|\mu| = r_l + t_j} \left(b_\mu(x)\right)_{l,j} \xi^\mu \ .$$

We will suppose that for any fixed x the elements of b_μ are in the same field $\mathbb{K}$ as the elements of a_μ. Note that when we pass to the involutive form we do not need any weights for the operator itself. However, we still need weights for the boundary operator.

REMARK 3.1. *Usually in the definition of (weighted) symbols one replaces the derivative ∂^μ in the original operator by $i^{|\mu|}\xi^\mu$. However, for simplicity of notation we delete the imaginary factor because it turns out that*

(i) it is not needed in the definition of ellipticity because the characteristic polynomial is homogeneous.

(ii) it is not necessary in the criteria for the Shapiro-Lopatinskij condition either. We will explain this in an appropriate place.

4. *Square DN-elliptic boundary problems*

In this section we will treat the square DN–elliptic case and consider the overdetermined case later. We will assume that the boundary operator has ν rows where 2ν is the degree of the characteristic polynomial. The reason for choosing ν conditions is that if there were less boundary conditions the problem would be underdetermined and if there were more conditions the boundary conditions should satisfy some compatibility conditions. The latter case is a special case of the more general case considered in section 6.

The relevant condition which guarantees the well-posedness of the boundary problem is called the *Shapiro-Lopatinskij condition* or the *complementing condition*; from now on we will abbreviate this as the SL–condition.

Let Ω be a smooth manifold with boundary and let Γ be its boundary. Let us fix some $p \in \Gamma$. Then there is some neighborhood U of p and local coordinates on U such that $U \cap \Omega$ is given in these coordinates by the half space $x_n \geqslant 0$ and $U \cap \Gamma$ is given by $x_n = 0$. From now on we will always work with these local coordinates and write $x = (x', x_n)$ where $x' = (x_1, \ldots, x_{n-1})$.

Now fixing some boundary point $(x', 0) \in \Gamma$ and some $\xi' \neq 0$ we consider the following ordinary differential operators with constant coefficients:

$$\begin{cases} \mathsf{A}_w(x', 0, \xi', D_n)u(x_n) = 0\ ,\ x_n > 0, \\ \mathsf{B}_w(x', 0, \xi', D_n)u(x_n)|_{x_n=0} = d, \end{cases} \tag{4.1}$$

where $d \in \mathbb{C}^\nu$, $D_n = \partial^{\mathbf{1}_n}/i$ and i is the imaginary unit. Hence A and B are some linear operators and after fixing some x' and ξ' we may interprete them as

$$\mathsf{A}_w : C^\infty(\mathbb{R} \times \mathbb{C}^m) \to C^\infty(\mathbb{R} \times \mathbb{C}^m)$$
$$\mathsf{B}_w : C^\infty(\mathbb{R} \times \mathbb{C}^m) \to \mathbb{C}^\nu$$

LEMMA 4.1 ([**3**]). *Suppose that A is DN–elliptic. Then*

$$\dim \big(\ker(\mathsf{A}_w(\xi', D_n)) \cap \mathfrak{M}_+ \big) = \nu.$$

DEFINITION 4.1. *The boundary operator (3.2) satisfies the SL–condition, if the initial value problem (4.1) has unique solution in $\mathfrak{M}_+$ for all d and for all $\xi' \neq 0$.*

Consider a boundary problem

$$\begin{cases} Ay = f, & x \in \Omega, \\ By = g, & x \in \Gamma. \end{cases} \tag{4.2}$$

DEFINITION 4.2. *The square boundary problem (4.2) is* DN-elliptic, *if*

(i) the operator A is DN-elliptic;

(ii) the operator B satisfies the SL–condition.

Now it turns out that these conditions guarantee that the boundary problem is well-posed in the following sense. Let us consider a BV–problem $\Phi = (A, B)$ with weight vectors s, t and r. Let us also choose some a such that

$$a \geqslant -\min t_j, \quad a \geqslant \max s_i, \quad a > \max r_i + 1/2. \tag{4.3}$$

Then one can show that Φ is a bounded operator if we choose the following Sobolev spaces [**4**]:

$$\Phi = (A, B) : H_{a+t}(\mathcal{E}_0) \to H_{a-s}(\mathcal{E}_1) \times H_{a-r-1/2}(\mathcal{G}_1) \tag{4.4}$$

Moreover one has the following result.

THEOREM 4.1 ([**4**]). *Let conditions* (4.3) *be satisfied. The following statements are equivalent:*

(i) the boundary problem (4.2) *is DN-elliptic;*

(ii) the operator Φ in (4.4) *is Fredholm and the following a priori estimate holds*

$$||y||_{a+t} \leqslant C\big(||f||_{a-s} + ||g||_{a-r-1/2} + ||y||_0\big).$$

If the solution is unique, the last term on the right can be omitted.

Recall that an operator is Fredholm, if it has a finite dimensional kernel and cokernel, and its image is closed.

4.1. *Algebraic formulation of the Shapiro-Lopatinskij condition*

It is evident that in general it is very hard to check the SL–condition using directly the definition. Agmon, Douglis and Nirenberg [**2**], [**3**] proposed a criterion to simplify the verification of the SL–condition, see also [**4**] for a discussion.

Consider a DN-elliptic square operator A. At the fixed boundary point $(x', 0) \in \Gamma$ we write for simplicity A_w and B_w instead of $\mathsf{A}_w(x', 0, \xi', \zeta)$ and $\mathsf{B}_w(x', 0, \xi', \zeta)$ when the meaning is clear from the context.

Let us fix some $\xi' \neq 0$ and let $\zeta_1, \ldots, \zeta_\nu$ be all the roots of the characteristic polynomial $p_{\mathsf{A}} \in \mathbb{R}[\zeta]$ lying in the upper half of the complex plane. We set

$$p_{\mathsf{A}}^+ = (\zeta - \zeta_1) \cdots (\zeta - \zeta_\nu) = \sum_{j=0}^{\nu} b_j \zeta^j.$$

and introduce the polynomials p_l^+, $l = 1, \dots, \nu$, as in (2.2).

The traditional formulation of a criterion which implies the SL–condition is as follows.

THEOREM 4.2 ([**3**], [**4**]). *The SL–condition is satisfied at a fixed boundary point if and only if for all $\xi' \neq 0$ the rows of the matrix*

$$\mathsf{B}_w \,\mathsf{adj}\,(\mathsf{A}_w)$$

are linearly independent modulo the polynomial p_{A}^+.

REMARK 4.1. *This criterion was originally formulated using symbol matrices with imaginary units, but that formulation is equivalent to our formulation. To see this let us first note that the polynomial p_{A}^+ does not depend on this choice. Then dividing all elements of $\mathsf{B}_w \,\mathsf{adj}\,(\mathsf{A}_w)$ by p_{A}^+ it is seen that the rank of matrix of remainders does not depend on this choice either.*

Now we would like to improve this criterion. To this end we need to formulate some preliminary results.

LEMMA 4.2. *There is an element $(\mathsf{adj}(\mathsf{A}_w))_{i,j}$ of $\mathsf{adj}(\mathsf{A}_w)$ which is not divisible by the polynomial p_{A}^+.*

Proof. Suppose that $\mathsf{adj}(\mathsf{A}_w) = p_{\mathsf{A}}^+ C$. On the other hand, by the property of adjoint matrix we get $\det(\mathsf{adj}(\mathsf{A}_w)) = p_{\mathsf{A}}^{m-1}$. Thus, $(p_{\mathsf{A}}^+)^m \det(C) = p_{\mathsf{A}}^{m-1}$. But this contradicts the definition of p_{A}^+.

□

Let us denote by v some column of the matrix $\mathsf{adj}(\mathsf{A}_w)$ which is nonzero modulo p_{A}^+. We set

$$\omega^l(x_n) = \frac{1}{2\pi i} \oint_{\gamma_+} \frac{v(\zeta) p_l^+(\zeta) e^{i\zeta x_n}}{p_{\mathsf{A}}^+(\zeta)}\, d\zeta, \quad l = 1, \dots, \nu.$$

where γ_+ is a simple closed curve oriented counterclockwise in the upper half of the complex plane surrounding all the roots of the polynomial p_{A}^+ and let ω be a matrix with columns ω^l.

LEMMA 4.3. *The columns of ω are a basis of the space $\ker(\mathsf{A}_w(D_n)) \cap \mathfrak{M}_+$.*

Proof. First note that by the property of adjoint matrix all elements of the vector $\mathsf{A}_w v$ are divisible by the polynomial p_{A}^+. Thus by the Cauchy integral theorem we get $\mathsf{A}_w(D_n) w^l(x_n) = 0$. Since $\Re(i\zeta) < 0$, we have that the columns of w belong to $\ker(\mathsf{A}_w(D_n)) \cap \mathfrak{M}_+$. Lemma 2.4 implies that the vectors $\omega^l(x_n)$, $l = 1, \dots, \nu$, are linearly independent. From Lemma 4.1 we have that $\dim\big(\ker(\mathsf{A}_w(D_n)) \cap \mathfrak{M}_+\big) = \nu$. As we have ν linearly independent vectors, then they are the basis of $\ker(\mathsf{A}_w(D_n)) \cap \mathfrak{M}_+$. □

Let us now consider the vector $h = \mathsf{B}_w v$. Dividing each element of h by the polynomial p_{A}^+ we get

$$h = q\, p_{\mathsf{A}}^+ + \mathsf{h} \qquad \text{where} \qquad \mathsf{h} = \sum_{\tau=0}^{\nu-1} \mathsf{h}^\tau \zeta^\tau \tag{4.5}$$

Let us introduce the matrix $\mathcal{H}$

$$\mathcal{H} = \big(\mathsf{h}^0, \dots, \mathsf{h}^{\nu-1}\big). \tag{4.6}$$

LEMMA 4.4. *The following equality holds*

$$\mathsf{B}_w(D_n)\omega(x_n)|_{x_n=0} = \mathcal{H}$$

Proof. Using Lemma 2.3 we have

$$\mathsf{B}_w(D_n)\omega^l(x_n)|_{x_n=0} = \frac{1}{2\pi i} \oint_{\gamma_+} \frac{\mathsf{B}_w(\zeta) v(\zeta) p_l^+(\zeta)}{p_{\mathsf{A}}^+(\zeta)}\, d\zeta =$$

$$\sum_{\tau=0}^{\nu-1} \mathsf{h}^\tau \frac{1}{2\pi i} \oint_{\gamma_+} \frac{\zeta^\tau p_l^+(\zeta)}{p_{\mathsf{A}}^+(\zeta)} d\zeta = \mathsf{h}^{l-1}, \quad l = 1, \dots, \nu.$$

□

Finally, we get the following improvement of the algebraic criterion for checking the SL-condition.

THEOREM 4.3. *Following statements are equivalent*

(SL) *the SL–condition is satisfied at a fixed boundary point.*

(SL1) *for any* $\xi' \neq 0$*, there is a column* v *of* $\mathsf{adj}(\mathsf{A}_w)$ *that is nonzero modulo the polynomial* p_{A}^{+} *such that the elements of the vector* $h = \mathsf{B}_w v$ *are linearly independent modulo* p_{A}^{+}.

(SL2) *for any* $\xi' \neq 0$*, there is a column* v *of* $\mathsf{adj}(\mathsf{A}_w)$ *that is nonzero modulo the polynomial* p_{A}^{+} *such that the following condition is fulfilled: there are no numbers* c_i *with at least one of them nonzero such that*

$$c_1 \mathsf{h}_1 + \cdots + c_\nu \mathsf{h}_\nu = 0$$

where h_i*,* $i = 1, \ldots, \nu$*, are the elements of* h *defined in* (4.5).

(SL3) *for any* $\xi' \neq 0$*, there is a column* v *of* $\mathsf{adj}(\mathsf{A}_w)$ *that is nonzero modulo the polynomial* p_{A}^{+} *such that* $\mathrm{rank}(\mathcal{H}) = \nu$ *where* $\mathcal{H}$ *is defined by* (4.6).

Proof. The conditions (SL1), (SL2) and (SL3) are easily seen to be equivalent.

(SL3) implies (SL). Let us suppose that the condition (SL3) is fulfilled and fix some $\xi' \neq 0$. Let v be a nonzero modulo p_{A}^{+} column of $\mathsf{adj}(\mathsf{A}_w)$ such that $\mathrm{rank}(\mathcal{H}) = \nu$. Since we have ν boundary conditions, we get $\mathrm{rank}(\mathcal{H}) = \mathrm{rank}(\mathcal{H}, d) = \nu$ for any d. Thus, the system $\mathcal{H}c = d$ has a solution. Using this vector c we construct the following function $u(x_n) = \omega(x_n)c$. Lemma 4.3 implies that $u \in \ker(\mathsf{A}_w(D_n)) \cap \mathfrak{M}_+$. By Lemma 4.4 we get

$$\mathsf{B}_w(D_n)u(x_n)|_{x_n=0} = \mathcal{H}c = d.$$

So the solution of the problem (4.1) exists.

Let us now prove the uniqueness of solution of (4.1). Suppose that there is a nonzero solution $u \in \ker(\mathsf{A}_w(D_n)) \cap \mathfrak{M}_+$ which satisfies $\mathsf{B}_w(D_n)u(x_n)|_{x_n=0} = 0$. Since the columns of ω are the basis of $\ker(\mathsf{A}_w(D_n)) \cap \mathfrak{M}_+$ there is a $c \neq 0$ such that $u(x_n) = \omega(x_n)c$. Then $\mathsf{B}_w(D_n)u(x_n)|_{x_n=0} = \mathcal{H}c = 0$. But this contradicts (SL3).

(SL) implies (SL3). Suppose now that the SL-condition is fulfilled. The SL-condition is equivalent to the following condition: for all $\xi' \neq 0$

$$\ker(\mathsf{B}_w) = \{0\}, \quad \mathrm{im}(\mathsf{B}_w) = \mathbb{C}^\nu,$$

where $\mathsf{B}_w : \ker(\mathsf{A}_w(D_n)) \cap \mathfrak{M}_+ \to \mathbb{C}^\nu$ is a linear operator. Fix some $\xi' \neq 0$ and let v be a nonzero modulo p_{A}^{+} column of $\mathsf{adj}(\mathsf{A}_w)$. Then according to Lemma 4.3 the columns of $\omega(x_n)$ are a basis of space $\ker(\mathsf{A}_w(D_n)) \cap \mathfrak{M}_+$. Since B_w is a linear operator, $\ker(\mathsf{B}_w) = \{0\}$ and $\dim\big(\mathrm{im}(\mathsf{B}_w)\big) = \nu$, we get that the columns of $\mathsf{B}_w(D_n)\omega(x_n)|_{x_n=0} = \mathcal{H}$ are a basis of the space $\mathrm{im}(\mathsf{B}_w)$. Hence, $\mathrm{rank}(\mathcal{H}) = \nu$.

□

REMARK 4.2. *Note that if there is* one *column* v *of* $\mathsf{adj}(\mathsf{A}_w)$ *that is nonzero modulo the polynomial* p_{A}^{+} *such that* $\mathrm{rank}(\mathcal{H}) = \nu$ *then this property is true for* any *column of* $\mathsf{adj}(\mathsf{A}_w)$ *that is nonzero modulo* p_{A}^{+}.

REMARK 4.3. *The difference between our formulation and the result of Agmon, Douglis and Nirenberg [**3**] is that they considered the whole adjoint in the criterion while it is only necessary to take one column which is nonzero modulo* p_{A}^{+}.

4.2. *Computational test for checking the SL-condition in two variable case*

The idea of the formulations of the criterion for the SL-condition is to consider ξ' as a "parameter" (to fix it) and not as a "variable". Hence in principle we should use the test for each ξ' separately. However, we would like to have a test which is valid for all ξ'. Unfortunately we have found such a test only in case of 2 independent variables. Note that this case is already very important in PDE theory.

The main difficulty is to compute p_{A}^{+}. In 2 variable case we can use the fact that p_{A} is homogeneous to factor it, but this idea does not work in case of more variables. For example if $p_{\mathsf{A}} = \xi_1^2 + \zeta^2$ then we can easily factor this using the factors of $\zeta^2 + 1$, so the problem reduces to the factorisation of the polynomial of one variable. However, if $p_{\mathsf{A}} = \xi_1^2 + \xi_2^2 + \zeta^2$ then we cannot proceed in the same way, even though formally we can factor this as $p_{\mathsf{A}} = (\zeta - i\sqrt{\xi_1^2 + \xi_2^2})(\zeta + i\sqrt{\xi_1^2 + \xi_2^2})$.

Consider the system (2.11) in two variables. Then the characteristic polynomial p_{A} is a homogeneous polynomial in two variables $\xi = (\xi_1, \zeta)$ of degree 2ν. Dehomogenising p_{A}, i.e. setting $\xi_1 = 1$, we get a polynomial $\hat{p}_{\mathsf{A}}$. In the same way we denote the dehomogenised symbol matrices of A_w and B_w by $\hat{\mathsf{A}}_w$ and $\hat{\mathsf{B}}_w$. It is easy to see that $\det(\hat{\mathsf{A}}_w) = \hat{p}_{\mathsf{A}}$.

Let $\mathbb{K}(a)$ be a splitting field for $\hat{p}_{\mathsf{A}}$ with minimal polynomial p_{min} of degree ℓ. Hence $\mathbb{K}(a)$ is a $\mathbb{K}$-vector space of dimension ℓ with basis $\{1, a, a^2, \dots, a^{\ell-1}\}$. Let $\hat{a} \in \mathbb{C}$ be a root of the polynomial p_{min}. It is easily seen that the map

$$\iota : \mathbb{K}(a) \to \mathbb{C}, \quad \iota\Big(\sum_{i=0}^{\ell-1} c_i a^i\Big) = \sum_{i=0}^{\ell-1} c_i \hat{a}^i, \tag{4.7}$$

is an injective homomorphism and it induces the ring homomorphism

$$\tilde{\iota} : \mathbb{K}(a)[\zeta] \to \mathbb{C}[\zeta], \quad \tilde{\iota}\Big(\sum_{i=0}^{\tau-1} b_i \zeta^i\Big) = \sum_{i=0}^{\tau-1} \iota(b_i)\zeta^i. \tag{4.8}$$

Also $\tilde{\iota}$ is injective and it preserves the degree of a polynomial. The notation $\iota(C)$ (resp. $\tilde{\iota}(C)$) where C is a matrix means that we apply the map ι (resp. $\tilde{\iota}$) to each element of C.

Let $\rho_1, \dots, \rho_\nu \in \mathbb{K}(a)$ be the roots of $\hat{p}_{\mathsf{A}}$ such that $\iota(\rho_1), \dots, \iota(\rho_\nu)$ are in the upper half of the complex plane and let

$$\hat{p}_{\mathsf{A}}^+ = (\zeta - \rho_1)\cdots(\zeta - \rho_\nu) \in \mathbb{K}(a)[\zeta]. \tag{4.9}$$

Further let

$$\tilde{p}_{\mathsf{A}}^+ = (\zeta - \iota(\rho_1)\xi_1)\cdots(\zeta - \iota(\rho_\nu)\xi_1), \quad \tilde{p}_{\mathsf{A}}^- = (\zeta - \overline{\iota(\rho_1)}\xi_1)\cdots(\zeta - \overline{\iota(\rho_\nu)}\xi_1) \tag{4.10}$$

Then evidently $p_{\mathsf{A}} = \tilde{p}_{\mathsf{A}}^+ \tilde{p}_{\mathsf{A}}^-$ and for a fixed $\xi' \neq 0$,

$$p_{\mathsf{A}}^+ = \begin{cases} \tilde{p}_{\mathsf{A}}^+ & , \text{ if } \xi_1 > 0 \\ \tilde{p}_{\mathsf{A}}^- & , \text{ if } \xi_1 < 0 \end{cases}$$

We will show that it is in fact sufficient to work with $\hat{p}_{\mathsf{A}}^+$. To this end we will need the following fact. For divisions in $\mathbb{C}[\xi_1, \zeta]$ we use lexicographic ordering with $\zeta > \xi_1$.

LEMMA 4.5. *There is an element $\hat{d}$ of $\mathsf{adj}(\hat{\mathsf{A}}_w)$ such that*

1. *$\hat{d}$ is not divisible by $\hat{p}_{\mathsf{A}}^+$;*
2. *$\hat{d}$ is not divisible by $\tilde{\iota}(\hat{p}_{\mathsf{A}}^+)$;*
3. *the corresponding element d of $\mathsf{adj}(\mathsf{A}_w)$ is not divisible by $\tilde{p}_{\mathsf{A}}^+$;*
4. *for any fixed $\xi_1 \neq 0$ the corresponding element d of $\mathsf{adj}(\mathsf{A}_w)$ is not divisible by p_{A}^+.*

Proof. Lemma 4.2 immediately implies the first statement. Hence there are polynomials $q,\ r \in \mathbb{K}(a)[\zeta]$ such that $\hat{d} = q\hat{p}_{\mathsf{A}}^+ + r$, $r \neq 0$ and $\deg r < \nu$, and consequnetly $\hat{d} = \tilde{\iota}(\hat{d}) = \tilde{\iota}(q)\tilde{\iota}(\hat{p}_{\mathsf{A}}^+) + \tilde{\iota}(r)$. Since $\tilde{\iota}$ is injective and it preserves the degree of a polynomial, $\tilde{\iota}(r)$ is a nonzero remainder which proves the second statement.

Then let us divide d by $\tilde{p}_{\mathsf{A}}^+$ which yields $d = q\tilde{p}_{\mathsf{A}}^+ + r$. But this is simply a homogenised version of $\hat{d} = \tilde{\iota}(q)\tilde{\iota}(\hat{p}_{\mathsf{A}}^+) + \tilde{\iota}(r)$. Hence r cannot be zero which proves the third statement.

Since d is not divisible by $\tilde{p}_{\mathsf{A}}^+$, there are polynomials $q,\ r \in \mathbb{C}[\xi_1, \zeta]$ such that $d = q\tilde{p}_{\mathsf{A}}^+ + r$ where $r \neq 0$ and none of the monomials of r is divisible by ζ^ν. Since r is homogeneous, it will remain nonzero, if we substitute some nonzero real number for ξ_1. Hence the fourth statement is valid for $\xi_1 > 0$ because in that case $p_{\mathsf{A}}^+ = \tilde{p}_{\mathsf{A}}^+$. Then if $\xi_1 < 0$, it is easy to see that $d = \bar{q}p_{\mathsf{A}}^+ + \bar{r}$. Hence in this case also the remainder $\bar{r}$ is nonzero. □

Suppose that $\hat{d}$ is in the j'th column of $\mathsf{adj}(\mathsf{A}_w)$. We denote by $\hat{v}$ (resp. v) the j'th column of $\mathsf{adj}(\hat{\mathsf{A}}_w)$ (resp. $\mathsf{adj}(\mathsf{A}_w)$).

Then dividing all elements of $\hat{h} = \hat{\mathsf{B}}_w\hat{v}$ by $\hat{p}_{\mathsf{A}}^+$ gives $\hat{h} = \hat{q}\hat{p}_{\mathsf{A}}^+ + \hat{\mathsf{h}}$ where $\deg \hat{\mathsf{h}}_i < \nu$. Setting

$$\hat{\mathsf{h}} = \sum_{\tau=0}^{\nu-1} \hat{\mathsf{h}}^\tau \zeta^\tau \tag{4.11}$$

we introduce the matrix

$$\hat{\mathcal{H}} = \big(\hat{\mathsf{h}}^0, \dots, \hat{\mathsf{h}}^{\nu-1}\big) \tag{4.12}$$

whose elements are in $\mathbb{K}(a)$. In the same way we can divide $\hat{h}$ by $\tilde{\iota}(\hat{p}_{\mathsf{A}}^+)$ and construct the analog of the matrix $\hat{\mathcal{H}}$. It is clear that the result will be $\iota(\hat{\mathcal{H}})$.

Set $\tilde{h} = \mathsf{B}_w v$. The i'th component of $\tilde{h}$ is a homogeneous polynomial in $\xi = (\xi', \zeta)$ of degree $r_i + 2\nu - s_j$ (v is the j'th column of $\mathsf{adj}(\mathsf{A}_w)$). Using the division algorithm we get $\tilde{h} = \tilde{q}\tilde{p}_{\mathsf{A}}^{+} + \tilde{\mathsf{h}}$. Since the components of $\tilde{q}$ and $\tilde{h}$ are homogeneous polynomials, the i'th component of $\tilde{\mathsf{h}}$ is also a homogeneous polynomial of the degree $r_i + 2\nu - s_j$. Setting now

$$\tilde{\mathsf{h}} = \sum_{\tau=0}^{\nu-1} \tilde{\mathsf{h}}^\tau \zeta^\tau \qquad \text{and} \qquad \tilde{\mathcal{H}} = \left(\tilde{\mathsf{h}}^0, \dots, \tilde{\mathsf{h}}^{\nu-1}\right)$$

we readily see that $\tilde{\mathsf{h}}_i^\tau = \iota(\hat{\mathsf{h}}_i^\tau)\xi_1^{r_i+2\nu-s_j-\tau}$ and that

$$\det(\tilde{\mathcal{H}}) = \xi_1^{\nu(2\nu-s_j-(\nu-1)/2)+r_1+\cdots+r_\nu} \det\left(\iota(\hat{\mathcal{H}})\right). \tag{4.13}$$

Finally we compute $h = \mathsf{B}_w v$ with some fixed $\xi_1 \neq 0$. Dividing h by p_{A}^{+} and constructing the corresponding matrix $\mathcal{H}$ as above we get for any fixed $\xi_1 \neq 0$,

$$\mathcal{H} = \begin{cases} \tilde{\mathcal{H}} & , \text{ if } \xi_1 > 0 \\ \overline{\tilde{\mathcal{H}}} & , \text{ if } \xi_1 < 0. \end{cases} \tag{4.14}$$

Now we can formulate a computational test for checking the SL-condition.

THEOREM 4.4. *An operator (A, B) satisfies the SL–condition if and only if* $\det(\hat{\mathcal{H}}) \neq 0$ *where $\hat{\mathcal{H}}$ is a matrix defined in* (4.12)*.*

Proof. The condition (SL3) of Theorem 4.3 is equivalent to $\det(\mathcal{H}) \neq 0$ for all $\xi_1 \neq 0$. By (4.14) this is equivalent to $\det(\tilde{\mathcal{H}}) \neq 0$ for all $\xi_1 \neq 0$. Using (4.13) we get the condition $\det\left(\iota(\hat{\mathcal{H}})\right) \neq 0$. But since ι is injective homomorphism this is equivalent to $\det(\hat{\mathcal{H}}) \neq 0$. □

Note that this formulation of the SL–condition can effectively be tested using Gröbner basis techniques, for example with the program Singular [**13**]. (See Appendices 8.1, 8.2, 8.3 for the appropriate commands in Singular).

EXAMPLE 4.1. *Let us consider the following system*

$$Ay = \begin{cases} y^1_{20} + y^1_{02} + y^3_{20} = 0, \\ 2y^1_{10} + y^2_{01} = 0, \\ y^2_{10} + y^3_{01} = 0 \end{cases} \qquad \textit{in} \quad \mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x_2 > 0\}. \tag{4.15}$$

Taking weights $t_1 = t_2 = t_3 = 2$ and $s_1 = 0$, $s_2 = s_3 = -1$, we get the weighted principal symbol of (4.15)

$$\mathsf{A}_w = \begin{pmatrix} \zeta^2 + \xi_1^2 & 0 & \xi_1^2 \\ 2\xi_1 & \zeta & 0 \\ 0 & \xi_1 & \zeta \end{pmatrix}.$$

Since $p_{\mathsf{A}} = \det(\mathsf{A}_w) = \zeta^4 + \zeta^2\xi_1^2 + 2\xi_1^4$, the system (4.15) *is DN-elliptic. We set $\hat{p}_{\mathsf{A}} = \zeta^4 + \zeta^2 + 2$.*

Computing with Singular *as discussed above, we get the following:*

- *the splitting field of $\hat{p}_{\mathsf{A}}$ is $\mathbb{Q}(\alpha)$ with a minimal polynomial*

$$p_{min} = \alpha^8 + 10\alpha^6 + 5\alpha^4 - 100\alpha^2 + 2116;$$

- *the roots of the polynomial $\hat{p}_{\mathsf{A}}$ are*

$$\rho_1 = -\rho_2 = -\tfrac{7}{105432}\alpha^7 - \tfrac{265}{52716}\alpha^5 - \tfrac{3485}{105432}\alpha^3 - \tfrac{6883}{17572}\alpha;$$
$$\rho_3 = -\rho_4 = \tfrac{7}{52716}\alpha^7 + \tfrac{265}{26358}\alpha^5 + \tfrac{3485}{52716}\alpha^3 - \tfrac{1903}{8786}\alpha.$$

Now we find with Singular *numerically the roots of the polynomial p_{min} in $\mathbb{C}$:*

$$\alpha_1 = -\overline{\alpha_2} = -\alpha_3 = \overline{\alpha_4} \approx -0.676 + i\,2.935\,,$$
$$\alpha_5 = -\alpha_6 = \overline{\alpha_7} = -\overline{\alpha_8} \approx -2.028 + i\,0.978\,.$$

Substituting α_1 in the roots of polynomial $\hat{p}_{\mathsf{A}}$ we get

$$\rho_1(\alpha_1) = -\rho_2(\alpha_1) = -\overline{\rho_3(\alpha_1)} = \overline{\rho_4(\alpha_1)} \approx 0.676 - i\,0.978\,.$$

So, for the primitive element α_1 the roots ρ_2 and ρ_4 are in the upper half of the complex plane. Hence, we get

$$\hat{p}_{\mathsf{A}}^{+} = (\zeta - \rho_2)(\zeta - \rho_4) \in \mathbb{Q}(\alpha)[\zeta].$$

Let us compute the adjoint matrix of $\hat{\mathsf{A}}_w$. We get

$$\mathsf{adj}(\hat{\mathsf{A}}_w) = \begin{pmatrix} \zeta^2 & 1 & -\zeta \\ -2\zeta & \zeta^3 + \zeta & 2 \\ 2 & -\zeta^2 - 1 & \zeta^3 + \zeta \end{pmatrix}$$

Let $\hat{v}$ be the first column of $\mathsf{adj}(\hat{\mathsf{A}}_w)$; it is nonzero modulo $\hat{p}_{\mathsf{A}}^{+}$. We will carry out the SL–test symbolically in the polynomial ring $\mathbb{Q}(\alpha)[\zeta]$. Since $\deg \hat{p}_{\mathsf{A}} = 4$, we need to impose 2 boundary conditions on the system (4.15)*. First consider the following boundary conditions*

$$B_1 y = \begin{cases} y^1 + y^3 = 0 \\ y^2 = 0 \end{cases} \quad \text{on} \quad \partial\mathbb{R}^2_+.$$

So the principal symbol of B_1 is

$$\mathsf{B}_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and hence} \quad \hat{\mathsf{B}}_1 \hat{v} = \begin{pmatrix} \zeta^2 + 2 \\ -2\zeta \end{pmatrix}.$$

Reducing this vector with respect to $\hat{p}_{\mathsf{A}}^{+}$ we get

$$\hat{\mathsf{h}} = \begin{pmatrix} \beta_1 \zeta + \beta_0 \\ -2\zeta \end{pmatrix}, \quad \hat{\mathcal{H}} = \begin{pmatrix} \beta_0 & \beta_1 \\ 0 & -2 \end{pmatrix}$$

where

$$\beta_1 = -\tfrac{7}{105432}\alpha^7 - \tfrac{265}{52716}\alpha^5 - \tfrac{3485}{105432}\alpha^3 + \tfrac{10689}{17572}\alpha, \quad \beta_0 = \tfrac{1}{382}\alpha^6 + \tfrac{15}{764}\alpha^4 - \tfrac{123}{764}\alpha^2 + \tfrac{579}{382}.$$

Since $\det(\hat{\mathcal{H}}) = -2\beta_0 \neq 0$, B_1 satisfies the SL-condition.

Let us now impose the following boundary conditions on the system (4.15)*.*

$$B_2 y = \begin{cases} y^1 = 0 \\ y^2_{01} = 0 \end{cases}$$

Computing $(\hat{\mathsf{B}}_2)_w \hat{v}$ and reducing with respect to $\hat{p}_{\mathsf{A}}^{+}$ we have

$$\hat{\mathsf{h}} = \begin{pmatrix} \beta_1 \zeta + b_0 \\ -2\beta_1 \zeta - 2b_0 \end{pmatrix}, \quad \hat{\mathcal{H}} = \begin{pmatrix} b_0 & \beta_1 \\ -2b_0 & -2\beta_1 \end{pmatrix}$$

where $b_0 = \frac{1}{382}\alpha^6 + \frac{15}{764}\alpha^4 - \frac{123}{764}\alpha^2 - \frac{185}{382}$. Now $\det(\hat{\mathcal{H}}) = 0$ and hence B_2 does not satisfy the SL-condition.

5. *Overdetermined elliptic boundary problems (general theory)*

5.1. *Differential boundary operators*

Suppose that the system (2.11) is overdetermined ($k \geqslant m$). Since any linear overdetermined DN-elliptic system becomes elliptic during its completion to involutive form [**15**], without loss of generality we can assume that the system (2.11) is elliptic with the principal symbol A. Hence we consider now a boundary problem

$$\begin{cases} Ay = f, & x \in \Omega, \\ By = g, & x \in \Gamma, \end{cases} \tag{5.1}$$

where A is a q'th order elliptic differential operator and B is of size $\tilde{\nu} \times m$; we write this as $B = \big(B_1, \ldots, B_{\tilde{\nu}}\big)$, and denote the order of the operator B_i by r_i. We will always suppose that $\tilde{\nu} \geqslant \nu$ because otherwise the problem would not be Fredholm: the kernel of the BV–operator $\Phi_0 = (A, B)$ would be infinite dimensional.

Note that now we still need a nontrivial weight vector for B, but for s and t we can take $t_j = 0$ and $s_i = q$. Consequently it is natural to consider Φ_0 as operating in the following Sobolev spaces:

$$\Phi_0 = (A, B) \,:\, H_a(\mathcal{E}_0) \to H_{a-q}(\mathcal{E}_1) \times H_{a-r-1/2}(\mathcal{G}_1) \tag{5.2}$$

where a is some appropriately chosen number and $r = (r_1, \ldots, r_{\tilde{\nu}})$. To study the well-posedness of this boundary problem we need to construct a compatibility operator for Φ_0. It turns out that this can be done by using a certain equivalent first order system.

DEFINITION 5.1. *A differential operator* $A : \mathsf{S}(\mathcal{E}_0) \to \mathsf{S}(\mathcal{E}_1)$ *is called* normalised *if*

(i) *A is a first order operator;*

(ii) *A is involutive;*

(iii) *the principal symbol* $\mathsf{A} : T^*\Omega \otimes \mathcal{E}_0 \to \mathcal{E}_1$ *is surjective.*

Condition (iii) means that there are no (explicit or implicit) algebraic (i.e., non-differential) relations between dependent variables in the system. If such relations exist, then we may use them to reduce the number of dependent variables. Note also that the condition (iii) is equivalent to the surjectivity of the symbol matrix M_1. This follows from the formula (2.12) and diagram (2.10).

DEFINITION 5.2. *A DB-operator Φ (resp. BV-operator (A, B)) is* normalised *if Φ^{11} (resp. A) is normalised and $\gamma\Phi^{21}$ (resp. B) contains only differentiation in directions tangent to the boundary.*

So the idea is to replace the original BV–operator by an equivalent normalised operator, compute its compatibility operator, and then use this to construct the compatibility operator for the original operator. To this end we must recall some facts about compatibility operators and make more precise what is meant by equivalence. Now intuitively one thinks that equivalent systems should have same solution spaces. However, this would be rather difficult to define precisely in a useful way. Instead, the concept of equivalence is defined with help of certain maps between bundles. We will next describe how this is done.

5.2. *Compatibility*

Of course we are mostly interested in differential operators, but the notion of compatibility operator can be defined for any bundle map.

DEFINITION 5.3. *Let $\Phi_0 : \mathcal{F}_0 \to \mathcal{F}_1$ be a bundle map. The bundle map $\Phi_1 : \mathcal{F}_1 \to \mathcal{F}_2$ is a* compatibility map *for Φ_0 if*

1. $\Phi_1\Phi_0 = 0$
2. *for any bundle map $\tilde{\Phi}_1 : \mathcal{F}_1 \to \tilde{\mathcal{F}}_2$ such that $\tilde{\Phi}_1\Phi_0 = 0$, there is a bundle map $T : \mathcal{F}_2 \to \tilde{\mathcal{F}}_2$ such that $\tilde{\Phi}_1 = T\Phi_1$.*

This idea leads naturally to

DEFINITION 5.4. *A complex*

$$\mathcal{C} \quad : \quad 0 \longrightarrow \mathcal{F}_0 \xrightarrow{\Phi_0} \mathcal{F}_1 \xrightarrow{\Phi_1} \mathcal{F}_2 \xrightarrow{\Phi_2} \cdots$$

is called a compatibility complex *for Φ_0 if every map Φ_i for $i \geqslant 1$ is a compatibility map for Φ_{i-1}.*

It is rather straightforward to show that a compatibility operator exists for operators with constant coefficients. Let $\Omega \subset \mathbb{R}^n$ be open and let $A_0 : \mathsf{S}(\mathcal{E}_0) \to \mathsf{S}(\mathcal{E}_1)$ be a differential operator with constant coefficients where $\mathcal{E}_0 = \Omega \times \mathbb{R}^{k_0}$ and $\mathcal{E}_1 = \Omega \times \mathbb{R}^{k_1}$.

Let us consider the matrix $\tilde{\mathsf{A}}_0$, the *full symbol* of A_0:

$$\tilde{\mathsf{A}}_0 = \sum_{|\mu| \leqslant q} a_\mu \xi^\mu$$

Note that for our purposes we do not need to add the factor $i^{|\mu|}$. Denoting by $a^1, \ldots, a^{k_1}$ the rows of $\tilde{\mathsf{A}}_0$ we may construct a free resolution of the module $M_0 = \langle a^1, \ldots, a^{k_1} \rangle$ as in (2.1):

$$0 \longrightarrow \mathbb{A}^{k_r} \xrightarrow{\tilde{\mathsf{A}}_r^T} \mathbb{A}^{k_{r-1}} \longrightarrow \cdots \xrightarrow{\tilde{\mathsf{A}}_2^T} \mathbb{A}^{k_2} \xrightarrow{\tilde{\mathsf{A}}_1^T} \mathbb{A}^{k_1} \xrightarrow{\tilde{\mathsf{A}}_0^T} \mathbb{A}^{k_0} \longrightarrow \mathbb{A}^{k_0}/M_0 \longrightarrow 0$$

Let us denote by A_i the differential operator corresponding to the syzygy matrix $\tilde{\mathsf{A}}_i$. Now we say that a complex $\mathcal{C}$ consisting of trivial bundles $\mathcal{E}_i = \Omega \times \mathbb{R}^{k_i}$ and operators A_i is a *Hilbert complex*, if the operators A_i are associated to the syzygy matrices of the free resolution of $\mathbb{A}$-module M_0.

THEOREM 5.1. *[**25**, p. 31] Let $\mathcal{C}$ be a complex of differential operators with constant coefficients. $\mathcal{C}$ is a compatibility complex for A_0 if and only if $\mathcal{C}$ is a Hilbert complex associated with the $\mathbb{A}$-module M_0.*

Note that the compatibility complex which exists by above Theorem can be constructively computed using Gröbner basis techniques, see Appendix 8.4. In the general case we have

THEOREM 5.2. *Every sufficiently regular differential operator has a compatibility operator.*

See [**8**], [**19**] and [**25**] for more details. Note that in above Theorems there is no need to pass to the normalised operator. However, in case of BV–operators the construction of the compatibility operator is more involved. Here we suppose at the outset that the operator is normalised, and then discuss later how this leads to the compatibility operator in the general case.

So let us consider a normalised BV–operator $\Phi = (A, B) : \mathsf{S}(\mathcal{E}_0) \to \mathsf{S}(\mathcal{E}_1) \times \mathsf{S}(\mathcal{G}_1)$. We will suppose that A is sufficiently regular and denote the compatibility operator of A by A_1. Then we will need to define the tangent part of A, denoted by A^τ [**8**]. The embedding of the boundary Γ in Ω induces in the sections of the jet bundles a map $\mathsf{e}_0 : J^1(\mathcal{E}_0)|_\Gamma \to J^1(\mathcal{E}_0|_\Gamma)$ such that

$$\mathsf{e}_0((j^1 y)|_x) = j^1(\gamma y)(x), \quad x \in \Gamma,\ y \in \mathsf{S}(\mathcal{E}_0).$$

We define the bundle $\mathcal{E}_1^\tau \to \Gamma$ by

$$\mathcal{E}_1^\tau = \mathcal{E}_1|_\Gamma / (\mathcal{A}\ \ker(\mathsf{e}_0)).$$

Let $\mathsf{pr}^\tau : \mathcal{E}_1 \to \mathcal{E}_1^\tau$ be the projection. So we may define uniquely a map $\mathcal{A}^\tau$ by requiring that the following diagram commutes

$$\begin{array}{ccc} J^1(\mathcal{E}_0) & \xrightarrow{\ \mathcal{A}\ } & \mathcal{E}_1 \\ {\scriptstyle \mathsf{e}_0\gamma}\downarrow & & \downarrow{\scriptstyle \mathsf{pr}^\tau} \\ J^1(\mathcal{E}_0|_\Gamma) & \xrightarrow{\ \mathcal{A}^\tau\ } & \mathcal{E}_1^\tau \end{array}$$

DEFINITION 5.5. *The differential operator $A^\tau : \mathsf{S}(\mathcal{E}_0|_\Gamma) \to \mathsf{S}(\mathcal{E}_1^\tau)$ defined by $A^\tau = \mathcal{A}^\tau j^1|_\Gamma$ is called the* tangent part *of A.*

It can be shown that if A is normalised, then so is A^τ [**8**]. Let us denote the compatibility operator of A^τ by A_1^τ. Then we define a differential operator

$$\Phi^\tau : \mathsf{S}^\tau(\mathcal{E}_0|_\Gamma) \to \mathsf{S}^\tau(\mathcal{E}_1^\tau) \times \mathsf{S}(\mathcal{G}_1) \quad , \quad \Phi^\tau(y) = (A^\tau y, By). \tag{5.3}$$

Let Φ_1^τ be a compatibility operator for Φ^τ. Note that Φ_1^τ exists by Theorem 5.2, if Φ^τ is sufficiently regular. If this is the case we say that BV-operator Φ is *regular*.

Note that in general Φ_1^τ may always be written in the form $\Phi_1^\tau(f', g) = \big(A_1^\tau f', \Upsilon^\tau(f', g)\big)$ where Υ^τ does not contain relations only between the components of f'. Let us then finally define

$$\Phi_1 : \mathsf{S}(\mathcal{E}_1) \times \mathsf{S}(\mathcal{G}_1) \to \mathsf{S}(\mathcal{E}_2) \times \mathsf{S}(\mathcal{G}_2) \quad , \quad \Phi_1(f, g) = (A_1 f, \Upsilon^\tau(\mathsf{pr}^\tau f, g)) \tag{5.4}$$

We will need the following important result [**8**, p. 40].

THEOREM 5.3. *If $\Phi = (A, B)$ is a regular elliptic normalised BV-operator then the DB-operator Φ_1 defined by* (5.4) *is a compatibility operator for Φ.*

This gives a construction of the compatibility operator for the normalised BV–operator. Next we will indicate how this can be used to construct the compatibility operator in the general case.

5.3. *Equivalence of operators*

We start by defining rigorously what is meant by equivalence of operators and complexes [**25**].

DEFINITION 5.6. *Two complexes $\mathcal{C}$ and $\mathcal{C}'$ are* equivalent *if the following conditions are satisfied:*

1. *there are maps M_i and N_i such that the following diagram commutes for all i*

$$\begin{array}{ccc} \mathcal{F}_i & \xrightarrow{\Phi_i} & \mathcal{F}_{i+1} \\ M_i \downarrow \uparrow N_i & & M_{i+1} \downarrow \uparrow N_{i+1} \\ \mathcal{F}'_i & \xrightarrow{\Phi'_i} & \mathcal{F}'_{i+1} \end{array}$$

2. *there are maps Ψ_i and Ψ'_i such that for all i*

$$\Psi_i\Phi_i + \Phi_{i-1}\Psi_{i-1} = \mathsf{id} - N_iM_i$$
$$\Psi'_i\Phi'_i + \Phi'_{i-1}\Psi'_{i-1} = \mathsf{id} - M_iN_i$$

The equivalence defined above is sometimes called homotopical or cochain equivalence.

DEFINITION 5.7. *Bundle maps $\Phi : \mathcal{F}_0 \to \mathcal{F}_1$ and $\Phi' : \mathcal{F}'_0 \to \mathcal{F}'_1$ are* equivalent *if the complexes*

$$0 \longrightarrow \mathcal{F}_0 \xrightarrow{\Phi} \mathcal{F}_1$$

$$0 \longrightarrow \mathcal{F}'_0 \xrightarrow{\Phi'} \mathcal{F}'_1$$

are equivalent.

The concept of equivalence is important for our purposes because of the following Theorems.

THEOREM 5.4. *Every sufficiently regular operator A may be transformed in a finite number of steps into an equivalent normalised operator. Every DB-operator Φ (resp. BV-operator (A, B)) whose component Φ^{11} (resp. A) is sufficiently regular is equivalent to a normalised DB-operator (resp. BV-operator).*

Then if we know a compatibility operator for some operator, we can construct a compatibility operator for an equivalent operator as follows.

THEOREM 5.5. *Let Φ_0 and Φ'_0 be equivalent bundle maps. If there is a compatibility complex for Φ'_0, then there is also a compatibility complex for Φ_0. Moreover their compatibility complexes are equivalent.*

Proof. Here we just give the formula for constructing the first compatibility map since we will need it in the sequel. For the details of the proof we refer to [**25**]. We fix some maps M_i, N_i $(i = 0, 1)$,Ψ_0 and Ψ'_0 as in Definition 5.6. We set $\mathcal{F}_2 = \mathcal{F}'_2 \oplus \mathcal{F}_1$. Then if we know the compatibility map Φ'_1, the compatibility map Φ_1 is given by the formula

$$\Phi_1 = \big(\Phi'_1 M_1\big) \oplus \big(\mathsf{id} - N_1M_1 - \Phi_0\Psi_0\big). \tag{5.5}$$

□

Hence to construct a compatibility operator for the BV–operator $\Phi = (A, B)$ we have to perform the following steps.

(1) Construct the involutive form of A.
(2) Prolong the system (if necessary) until the order of the system is higher than the the order of normal derivatives in the boundary operator.
(3) Construct an equivalent first order system.
(4) Eliminate (if necessary) the extra variables using the algebraic relations in the system.
(5) Construct the compatibility operator for the normalised system with the formula (5.4).
(6) Construct the compatibility operator for the BV-operator using (5.5).

Next we will discuss how to implement these steps in practice.

5.4. *Constructions*

The first step is Cartan–Kuranishi algorithm and the second step is merely a simple differentiation. Since these steps are extensively discussed elsewhere, for example in [**22**], we will simply observe that they can effectively be performed for example with DETools package in MuPAD.

Step (3). *Equivalent first order operator.* This construction can be found in [**19**]. Let A_0 be a differential operator of order q such that the first two steps of the above construction are already performed and let A_0' be an equivalent first order operator. We want to write A_0 as $A_0 = \bar{A}\, j^{q-1}$ where $\bar{A}$ is a first order operator. We introduce new dependent variables for all derivatives of order less than or equal to $q-1$ and denote them by $z^{j,\mu}$, with $|\mu| \leqslant q-1$. The operator $\bar{A}$ is obtained from A by performing the following substitutions:[5]

$$y^j_\mu \longmapsto \begin{cases} z^{j,\mu} & \text{if } |\mu| \leqslant q-1\ , \\ \partial^{\mu_2} z^{j,\mu_1} & \text{if } |\mu| = q \text{ where } |\mu_1| = q-1 \text{ and } \mu_1+\mu_2=\mu\ . \end{cases} \tag{5.6}$$

The desired first order operator is $A_0'z = \big(\bar{A}z, \mathcal{D}^{q-1}z\big)$ where $\mathcal{D}^{q-1}$ is the compatibility operator for j^{q-1}. In coordinates it can be written as $\mathcal{D}^{q-1} = D_1^{q-1} \otimes I_m$ where D_1^{q-1} is given by

$$\partial_i z^{j,\mu} = \begin{cases} z^{j,\mu+\mathbf{1}_i} & 0 \leqslant |\mu| < q-1,\ 1 \leqslant i \leqslant n\ , \\ \partial_k z^{j,\mu-\mathbf{1}_k+\mathbf{1}_i} & |\mu| = q-1,\ i > \mathsf{cls}\,\mu,\ k = \mathsf{cls}\,\mu\ . \end{cases} \tag{5.7}$$

It is straightforward to verify that A_0 and A_0' are indeed equivalent by considering the following diagram.

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{E}_0 & \xrightarrow{\quad A_0 \quad} & \mathcal{E}_1 \\ & & {\scriptstyle j^{q-1}}\downarrow\ \uparrow{\scriptstyle \mathsf{pr}_0} & & {\scriptstyle \iota_1}\downarrow\ \uparrow{\scriptstyle \mathsf{pr}_1} \\ 0 & \longrightarrow & \mathcal{E}_0' & \xrightarrow{\quad A_0' \quad} & \mathcal{E}_1' \end{array}$$

Since $\mathcal{E}_0'$ is essentially $J_{q-1}(\mathcal{E}_0)$, pr_0 is simply the canonical projection, and ι_1 (resp. pr_1) is the obvious injection (resp. projection).

Step(4). *Normalization.* We have now constructed A_0' which is a first order operator, equivalent to A_0. Also the corresponding B_0' contains differentiations only in the directions tangent to the boundary. Now to eliminate algebraic relations in A_0' can be done simply by performing Gaussian elimination followed by appropriate back substitutions. We denote by A_0'' the resulting operator. Then we have the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{E}_0 & \xrightarrow{\quad A_0 \quad} & \mathcal{E}_1 & \xrightarrow{A_1} & \mathcal{E}_2 \simeq \mathcal{E}_2'' \oplus \mathcal{E}_1 \\ & & {\scriptstyle j^{q-1}}\downarrow\ \uparrow{\scriptstyle \mathsf{pr}_0} & & {\scriptstyle \iota_1}\downarrow\ \uparrow{\scriptstyle \mathsf{pr}_1} & & \\ 0 & \longrightarrow & \mathcal{E}_0' & \xrightarrow{\quad A_0' \quad} & \mathcal{E}_1' & \xrightarrow{\quad A_1' \quad} & \mathcal{E}_2' \\ & & {\scriptstyle \mathsf{pr}_0'}\downarrow\ \uparrow{\scriptstyle \iota_0'} & & {\scriptstyle \alpha}\downarrow\ \uparrow{\scriptstyle \iota_1'} & & \\ 0 & \longrightarrow & \mathcal{E}_0'' & \xrightarrow{\quad A_0'' \quad} & \mathcal{E}_1'' & \xrightarrow{\quad A_1'' \quad} & \mathcal{E}_2'' \end{array} \tag{5.8}$$

Here the definitions of pr_0', ι_0' and ι_1' are obvious, but α which describes the result of a Gaussian elimination does not have any easy explicit expression.

However, we are really interested in BV–operators. This leads to the diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{E}_0 & \xrightarrow{\quad (A,B) \quad} & \mathcal{E}_1 \times \mathcal{G}_1 & \xrightarrow{\Phi_1} & \mathcal{E}_2 \times \mathcal{G}_2 \\ & & {\scriptstyle j^{q-1}}\downarrow\ \uparrow{\scriptstyle \mathsf{pr}_0} & & {\scriptstyle \tilde{\iota}_1}\downarrow\ \uparrow{\scriptstyle \tilde{\mathsf{pr}}_1} & & \\ 0 & \longrightarrow & \mathcal{E}_0' & \xrightarrow{\quad (A',B') \quad} & \mathcal{E}_1' \times \mathcal{G}_1' & \xrightarrow{\Phi_1'} & \mathcal{E}_2' \times \mathcal{G}_2' \\ & & {\scriptstyle \mathsf{pr}_0'}\downarrow\ \uparrow{\scriptstyle \iota_0'} & & {\scriptstyle \beta}\downarrow\ \uparrow{\scriptstyle \tilde{\iota}_1'} & & \\ 0 & \longrightarrow & \mathcal{E}_0'' & \xrightarrow{\quad (A'',B'') \quad} & \mathcal{E}_1'' \times \mathcal{G}_1'' & \xrightarrow{\Phi_1''} & \mathcal{E}_2'' \times \mathcal{G}_2'' \end{array} \tag{5.9}$$

[5]Obviously there are many ways to perform such a substitution, as there are many ways to split the multi index μ into two parts. However, for our purposes any choice is fine.

Let us describe shortly various maps involved.

- The maps in the first column are the same as in the diagram (5.8).
- $\tilde{\iota}_1 = (\iota_1, \mathsf{id})$ where ι_1 is as in (5.8). In particular $\mathcal{G}_1 = \mathcal{G}_1'$.
- $\tilde{\mathsf{pr}}_1 = (\mathsf{pr}_1, \mathsf{id})$ where pr_1 is as in (5.8).
- $\tilde{\iota}_1' = (\iota_1', \hat{\iota}_1)$ where ι_1' is as in (5.8) and $\hat{\iota}_1$ is induced by ι_0'.
- The map β is a DB operator, induced by the Gaussian elimination. We will write it as

$$\beta(f', g') = \big(\alpha(f'), \gamma\beta^{21}(f') + \beta^{22}(g')\big)$$

where α is as in (5.8).

Step (5). *Compatibility operator for the normalised BV–operator.* Let $\Phi_0'' = (A_0'', B_0'')$ be a normalised BV–operator. We need to perform the following tasks.

5a Compute the tangent part of A_0''.
Let M be the module generated by the rows of A_0''. We choose a monomial product ordering such that ξ_n is bigger than all other ξ_i. Then we define a TOP module ordering using this monomial ordering and compute the Gröbner basis of M. Now by Lemma 2.1 $A_0^{''\tau}$ is defined by the elements of the Gröbner basis that do not contain ξ_n. See Appendix 8.5 for the appropriate computations with Singular.

5b Set $\Phi_0^{''\tau} = (A_0^{''\tau}, B_0'')$.

5c Compute $\Phi_1^{''\tau}$, the compatibility operator of $\Phi_0^{''\tau}$.
We choose a POT module ordering and compute $\Phi_1^{''\tau}$. Then Lemma 2.2 implies that we can now extract $\Upsilon^{''\tau}$ by simple inspection.

5d The compatibility operator Φ_1'' can now defined by the formula

$$\Phi_1''(f'', g'') = \big(A_1'' f'', \gamma\Phi_1^{21''} f'' + \Phi_1^{22''} g''\big) = \big(A_1'' f'', \Upsilon^{''\tau}(\mathsf{pr}^\tau f'', g'')\big)$$

Step (6). *Compatibility operator for the original involutive BV–operator.* Using diagram (5.9) we may rewrite the formula (5.5) for the compatibility operator Φ_1 of the operator $\Phi_0 = (A, B)$ as

$$\Phi_1 = \big(\Phi_1''\,\beta\,\tilde{\iota}_1\big) \oplus \big(\mathsf{id} - \tilde{\mathsf{pr}}_1\,\tilde{\iota}_1'\,\beta\,\tilde{\iota}_1 - (A, B)\Psi_0\big). \tag{5.10}$$

where Ψ_0 is determined by the equation

$$\Psi_0(A, B) = \mathsf{id} - \mathsf{pr}_0\iota_0'\mathsf{pr}_0' j^{q-1}.$$

If the system $Ay = 0$ does not contain algebraic relations between the dependent variables then we may choose $\Psi_0 = 0$. If this is not the case then we could as well apply Gaussian elimination to the original system and remove algebraic dependencies. Hence without loss of generality and for simplicity of notation we suppose in the sequel that $\Psi_0 = 0$. In this case we have

$$\Phi_1^{11} = A_1 = \big(A_1''\alpha\iota_1\big) \oplus \big(\mathsf{id} - \mathsf{pr}_1\iota_1'\alpha\iota_1\big),$$

where A_1 is the compatibility operator of A. The other parts of the compatibility operator are given by:

$$\Phi_1^{21} f + \Phi_1^{22} g = \big(\Phi_1^{''21}\alpha\iota_1(f) + \Phi_1^{''22}(\gamma\beta^{21}\iota_1(f) + \beta^{22}(g))\big) \oplus \big(g - \hat{\iota}_1(\gamma\beta^{21}\iota_1(f) + \beta^{22}(g))\big).$$

We will see below that for the purposes of this paper we are particularly interested in Φ_1^{22}. This is given by

$$\Phi_1^{22} = \big(\Phi_1^{''22}\beta^{22}\big) \oplus \big(\mathsf{id} - \hat{\iota}_1\beta^{22}\big). \tag{5.11}$$

Let us now consider an example of computation of the compatibility operator A_1 of the operator A_0 using the compatibility operator A_1'' of the equivalent normalised operator A_0'' and construction (5.5).

EXAMPLE 5.1. *Consider the following familiar stationary Stokes problem in 2 dimensions:*

$$A : \begin{cases} -\Delta u + \nabla p = 0, \\ \nabla \cdot u = 0 \end{cases} \qquad \text{in} \quad \mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x_2 > 0\}, \tag{5.12}$$

where $u = (u^1, u^2)$ is the velocity field and p is the pressure.

Completing it to the involutive form we get the following overdetermined system.

$$A_0: \quad \begin{cases} -\Delta u + \nabla p = 0, \\ -\Delta p = 0, \\ \nabla \cdot u = 0, \\ u^1_{20} + u^2_{11} = 0, \\ u^1_{11} + u^2_{02} = 0 \end{cases} \qquad \textit{in} \quad \mathbb{R}^2_+. \tag{5.13}$$

Let us go over to an equivalent normalised boundary value operator. Introducing 9 new variables

$$z^{1,00} = u^1, z^{2,00} = u^2, z^{3,00} = p,$$
$$z^{1,10} = u^1_{10}, z^{2,10} = u^2_{10}, z^{3,10} = p_{10},$$
$$z^{1,01} = u^1_{01}, z^{2,01} = u^2_{01}, z^{3,01} = p_{01},$$

as in (5.6) *and substituting them in* (5.13)*, and also adding the compatibility equations* (5.7)*, we get the first order operator, denoted by* A'_0*. The corresponding system is not normalised since there is an algebraic relation* $z^{1,10} + z^{2,01} = 0$ *between the dependent variables. Now using this relation we can exclude the variable* $z^{2,01}$ *from the system and obtain the normalised operator* A''_0*.*

$$A'_0: \quad \begin{cases} -z^{1,10}_{10} - z^{1,01}_{01} + z^{3,10} = 0, \\ -z^{2,10}_{10} - z^{2,01}_{01} + z^{3,01} = 0, \\ -z^{3,10}_{10} - z^{3,01}_{01} = 0, \\ z^{1,10} + z^{2,01} = 0, \\ z^{1,10}_{10} + z^{2,01}_{10} = 0, \\ z^{1,10}_{01} + z^{2,01}_{01} = 0, \\ z^{1,00}_{10} - z^{1,10} = 0, \\ z^{1,00}_{01} - z^{1,01} = 0, \\ z^{2,00}_{10} - z^{2,10} = 0, \\ z^{2,00}_{01} - z^{2,01} = 0, \\ z^{3,00}_{10} - z^{3,10} = 0, \\ z^{3,00}_{01} - z^{3,01} = 0, \\ z^{1,10}_{01} - z^{1,01}_{10} = 0, \\ z^{2,10}_{01} - z^{2,01}_{10} = 0, \\ z^{3,10}_{01} - z^{3,01}_{10} = 0. \end{cases} \qquad A''_0: \quad \begin{cases} -z^{1,10}_{10} + z^{3,10} - z^{1,01}_{01} = 0, \\ z^{1,10}_{01} - z^{2,10}_{10} + z^{3,01} = 0, \\ -z^{3,10}_{10} - z^{3,01}_{01} = 0, \\ z^{1,00}_{10} - z^{1,10} = 0, \\ z^{1,00}_{01} - z^{1,01} = 0, \\ z^{2,00}_{10} - z^{2,10} = 0, \\ z^{2,00}_{01} + z^{1,10} = 0, \\ z^{3,00}_{10} - z^{3,10} = 0, \\ z^{3,00}_{01} - z^{3,01} = 0, \\ z^{1,10}_{01} - z^{1,01}_{10} = 0, \\ z^{1,10}_{10} + z^{2,10}_{01} = 0, \\ z^{3,10}_{01} - z^{3,01}_{10} = 0. \end{cases}$$

In this example we have the following fiber dimension of the bundles:

$$\mathsf{dim}(\mathcal{E}_1) = 6, \quad \mathsf{dim}(\mathcal{E}'_1) = 15, \quad \mathsf{dim}(\mathcal{E}''_1) = 12.$$

To construct the compatibility operator A''_1 *for the normalised operator* A''_0 *we compute with* Singular *the syzygy module generated by the rows of the full symbol matrix of the operator* A''_0 *and get*

$$A''_1(f'') = \big(- f''^{4}_{01} + f''^{5}_{10} - f''^{10}, f''^{6}_{01} + f''^{7}_{10} - f''^{11}, \\ - f''^{8}_{01} + f''^{9}_{10} - f''^{12}, f''^{1}_{10} + f''^{2}_{01} + f''^{3} - f''^{10}_{01} + f''^{11}_{10} \big).$$

To construct the compatibility operator A_1 *for the operator* A_0 *let us construct maps from the diagram* (5.8)*. The map* $\alpha : \mathcal{E}'_1 \to \mathcal{E}''_1$ *is given by*

$$\alpha(f') = (f'^1, f'^2 + f'^6, f'^3, f'^7, f'^8, f'^9, f'^{10} + f'^4, f'^{11}, f'^{12}, f'^{13}, f'^{14} + f'^5, f'^{15}).$$

Combining this with the inclusion map $\iota_1(f) = (f, 0)$ *gives*

$$\alpha\iota_1(f) = \big(f^1, f^2 + f^6, f^3, 0, 0, 0, f^4, 0, 0, 0, f^5, 0\big),$$
$$A''_1\alpha\iota_1(f) = \big(0, f^4_{10} - f^5, 0, f^1_{10} + f^2_{01} + f^6_{01} + f^3 + f^5_{10}\big).$$

The inclusion map $\iota_1' : \mathcal{E}_1'' \to \mathcal{E}_1'$ is given by

$$\iota_1'(f'') = \big(f''^1, f''^2, f''^3, 0, 0, 0, f''^4, \dots, f''^{12}\big).$$

So we have

$$\iota_1'\alpha\iota_1(f) = \big(f^1, f^2 + f^6, f^3, 0, 0, 0, 0, 0, 0, f^4, 0, 0, 0, f^5, 0\big).$$

The map $\mathsf{pr}_1 : \mathcal{E}_1' \to \mathcal{E}_1$ is defined by

$$\begin{aligned}\mathsf{pr}_1(f') = \big(&f'^1 - f'^7_{10} - f'^8_{01} + f'^{11}, f'^2 - f'^9_{10} - f'^{10}_{01} + f'^{12},\\ &f'^3 - f'^{11}_{10} - f'^{12}_{01}, f'^4 + f'^7 + f'^{10}, f'^5 + f'^7_{10} + f'^{10}_{10}, f'^6 + f'^7_{01} + f'^{10}_{01}\big).\end{aligned}$$

Hence, we get

$$(\mathsf{id} - \mathsf{pr}_1\iota_1'\alpha\iota_1)(f) = \big(0, f^4_{01} - f^6, 0, 0, f^5 - f^4_{10}, f^6 - f^4_{01}\big).$$

Thus, (5.5) *gives the following compatibility operator A_1 of A_0*

$$A_1(f) = (f^4_{01} - f^6, f^4_{10} - f^5, f^1_{10} + f^2_{01} + f^3 + f^5_{10} + f^6_{01}).$$

Note that we can also compute the compatibility operator A_1 of the operator A_0 by computing with Singular the syzygy module of the module generated by the rows of the full symbol matrix of A_0. However, the above computations are needed when we consider the Stokes problem with boundary conditions in the next example. In particular we show how to compute the part Φ_1^{22} of the compatibility operator Φ_1 for $\Phi_0 = (A, B)$, since it is necessary for checking the SL-condition for overdetermined boundary problems.

EXAMPLE 5.2. *Let us consider the operator A_0 given by* (5.13)*. Then the tangent part of the equivalent normalised operator operator A_0'' is*

$$A_0''^{\tau} z = \big(z^{2,10}_{10} - z^{1,01}_{10} - z^{3,01}, -z^{3,10} + z^{3,00}_{10}, -z^{2,10} + z^{2,00}_{10}, -z^{1,10} + z^{1,00}_{10}\big).$$

Let us define the following boundary operator

$$B : \quad \begin{cases} u^1 + u^2_{01} - p_{10} = 0,\\ -p - 2u^2_{10} + p_{10} = 0,\\ u^1_{10} - u^2_{10} = 0,\\ -u^2_{10} + p_{10} = 0,\\ \nabla \cdot u = 0, \end{cases} \qquad \text{on} \quad \Gamma = \partial\mathbb{R}^2_+ \tag{5.14}$$

for the operator A_0. Then we get

$$B' : \quad \begin{cases} z^{1,00} + z^{2,01} - z^{3,10} = 0,\\ -z^{3,00} - 2z^{2,10} + z^{3,10} = 0,\\ z^{1,10} - z^{2,10} = 0,\\ -z^{2,10} + z^{3,10} = 0,\\ z^{1,10} + z^{2,01} = 0, \end{cases} \qquad B'' : \quad \begin{cases} z^{1,00} - z^{1,10} - z^{3,10} = 0,\\ -z^{3,00} - 2z^{2,10} + z^{3,10} = 0,\\ z^{1,10} - z^{2,10} = 0,\\ -z^{2,10} + z^{3,10} = 0. \end{cases}$$

In this example we have the following fiber dimension of the bundles:

$$\mathsf{dim}(\mathcal{G}_1) = 5, \quad \mathsf{dim}(\mathcal{G}_1') = 5, \quad \mathsf{dim}(\mathcal{G}_1'') = 4, \quad \mathsf{dim}(\mathcal{E}_1''^{\tau}) = 4.$$

Let us compute the compatibility operator $\Phi_1''^{\tau}$ for the operator $\Phi''^{\tau} z = (A_0''^{\tau} z, B'' z)$ defined on Γ. Computing with Singular *the syzygy module for the module generated by the rows of the full symbol matrix of the operator Φ''^{τ} we have*

$$(0, -2\xi_1 + 1, 0, -\xi_1 - 1, \xi_1^2 + \xi_1, -2\xi_1^2 + \xi_1, \xi_1^2 - 1, 3\xi_1^2 - 2\xi_1 + 1).$$

In this example we have that the compatibility operator $\Phi_1''^{\tau}$ is in fact Υ^{τ} and we have

$$\begin{aligned}\Phi_1''^{\tau}(f^{\tau}, g'') = &-2f^{2,\tau}_{10} + f^{2,\tau} - f^{4,\tau}_{10} - f^{4,\tau} +\\ &g''^1_{20} + g''^1_{10} - 2g''^2_{20} + g''^2_{10} + g''^3_{20} - g''^3 + 3g''^4_{20} - 2g''^4_{10} + g''^4 .\end{aligned}$$

The projection $\mathsf{pr}^\tau : \mathcal{E}_1'' \to \mathcal{E}_1^{''\tau}$ *is given by*

$$\mathsf{pr}^\tau(f'') = (f^{''10}|_\Gamma - f^{''2}|_\Gamma, f^{''8}|_\Gamma, f^{''6}|_\Gamma, f^{''4}|_\Gamma).$$

Thus, the compatibility operator Φ_1'' *for the normalised operator* (A_0'', B'') *is*

$$\Phi_1''(f'', g'') = \big(A_1''f'', -2f_{10}^{''8}|_\Gamma + f^{''8}|_\Gamma - f_{10}^{''4}|_\Gamma - f^{''4}|_\Gamma + g_{20}^{''1} + g_{10}^{''1} - 2g_{20}^{''2} + g_{10}^{''2} + g_{20}^{''3} - g^{''3} + 3g_{20}^{''4} - 2g_{10}^{''4} + g^{''4}\big)$$

which implies that

$$\Phi_1^{''22}(g'') = g_{20}^{''1} + g_{10}^{''1} - 2g_{20}^{''2} + g_{10}^{''2} + g_{20}^{''3} - g^{''3} + 3g_{20}^{''4} - 2g_{10}^{''4} + g^{''4}\ .$$

The map $\beta : \mathcal{E}_1' \times \mathcal{G}_1 \to \mathcal{E}_1'' \times \mathcal{G}_1''$ *is defined by*

$$\beta(f', g') = (\alpha(f'), g^{'1} - f^{'4}|_\Gamma, g^{'2}, g^{'3}, g^{'4}).$$

Hence we obtain

$$\beta^{22}(g') = \big(g^{'1}, g^{'2}, g^{'3}, g^{'4}\big).$$

So the formula (5.11) *yields*

$$\Phi_1^{22}g = (g_{20}^1 + g_{10}^1 - 2g_{20}^2 + g_{10}^2 + g_{20}^3 - g^3 + 3g_{20}^4 - 2g_{10}^4 + g^4, 0, 0, 0, 0, g^5). \tag{5.15}$$

6. *Overdetermined elliptic boundary problems (the SL-condition)*

6.1. *Well-posed problems for overdermined elliptic PDEs*

¿From now on let B_w be the weighted principal symbol of $B = (B_1, \dots, B_{\tilde\nu})$ with weights $t_j = 0$ for the dependent variables and r_i for the equations where r_i is the order of the operator B_i, $i = 1, \dots, \tilde\nu$. Let Φ_1 be a compatibility operator for (A, B). We denote by $(\Phi_1^{22})_w$ the weighted principal symbol of Φ_1^{22} with the weights $-r_j$ for dependent variables and the weights δ_i for equations. We denote by τ the number of rows of Φ_1^{22}.

Set $r = (r_1, \dots, r_{\tilde\nu})$ and $\delta = (\delta_1, \dots, \delta_\tau)$, and let $\mathcal{G}_1 = \bigoplus_{\ell=1}^{\tilde\nu} G_1^\ell$ and $\mathcal{G}_2 = \bigoplus_{\ell=1}^{\tau} G_2^\ell$ be the direct sum decompositions of the bundles $\mathcal{G}_1$ and $\mathcal{G}_2$.

DEFINITION 6.1. *An operator* (A, B) *satisfies the* SL–condition *if for any* $x \in \Gamma$ *and* $\xi' \neq 0$ *the complex*

$$0 \longrightarrow \ker(\mathsf{A}(x, \xi', D_n)) \cap \mathfrak{M}_+ \xrightarrow{\mathsf{B}_w(x,\xi',D_n)} \mathcal{G}_1|_x \xrightarrow{(\Phi_1^{22})_w(x,\xi')} \mathcal{G}_2|_x\ , \tag{6.1}$$

is exact.

Here the operator $\mathsf{B}_w(x, \xi', D_n)$ is interpreted as $\mathsf{B}_w(x, \xi', D_n)u(x_n)|_{x_n=0}$.

REMARK 6.1. *Perhaps it would be more appropriate to use the term* the generalized SL–condition *in the above definition. However, for simplicity of language we prefer to use the SL–condition even in this case.*

The basic theorem on the solvability of the boundary problem (5.1) is

THEOREM 6.1 ([**9**]). *Let* $(A, B) : C^\infty(\mathcal{E}_0) \to C^\infty(\mathcal{E}_1) \times C^\infty(\mathcal{G}_1)$ *be a regular boundary value problem operator, and suppose that* A *is involutive and elliptic,* (A, B) *satisfies the SL-condition and the number* $\dim\big(\ker(\mathsf{A}(x, \xi', D_n))\big)$ *does not depend on* (x, ξ')*. In this case, if all the mappings in the complex*

$$0 \longrightarrow H_s(\mathcal{E}_0) \xrightarrow{(A,B)} H_{s-q}(\mathcal{E}_1) \times H_{s-r-1/2}(\mathcal{G}_1) \xrightarrow{\Phi_1} H_{s-q-1}(\mathcal{E}_2) \times H_{s-\eta}(\mathcal{G}_2)$$

where η *is some appropriate vector, are bounded, then there is a number* s *such that its cohomologies are finite-dimensional and their dimensions remain invariant when* s *is replaced by* $s' > s$*.*

We would like to propose a computational test for checking the SL-condition. But first we need to consider some preliminary results.

6.2. *Reduction of an overdetermined system to an equivalent square upper triangular one*

Fixing some boundary point $(x', 0) \in \Gamma$ and some vector $\xi' \neq 0$ we consider an overdetermined ordinary differential system with constant coefficients

$$\mathsf{A}(\xi', D_n)u(x_n) = 0. \tag{6.2}$$

We will first reduce this system to the upper triangular system

$$\mathsf{U}(\xi', D_n)u(x_n) = \begin{pmatrix} * & * & \dots & * \\ & * & \dots & * \\ & & \ddots & \vdots \\ & & & * \\ & & & \vdots \\ & & & * \end{pmatrix} u(x_n) = 0 \tag{6.3}$$

where $\mathsf{U}_{ij} = 0$ if $i > j$ and $j < m$. Writing now ζ for D_n we may interpret the elements of A and U as elements of the ring $\mathbb{R}[\zeta]$.

THEOREM 6.2. *There is a matrix* $T \in \big(\mathbb{R}[\zeta]\big)^{k\times k}$ *such that* $T\mathsf{A} = \mathsf{U}$ *with* U *as in* (6.3) *and* $\det(T) = 1$.

Proof. Let a and b be two rows of A and C be a $2 \times m$ matrix whose first row is a and second row is b. Now we want to construct a matrix P such that PC has zero in position $(2, 1)$ and $\det(P) = 1$.

First note that if $b_1 = 0$ then we take $P = I$ and if $a_1 = 0$ then

$$P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Now let us suppose that $a_1 \neq 0$ and $b_1 \neq 0$. Let $h = \mathsf{gcd}(a_1, b_1)$ be the greatest common divisor of a_1 and b_1. Then we have $a_1 = -hp_{22}$ and $b_1 = hp_{21}$ for some polynomials p_{21} and p_{22} such that $\mathsf{gcd}(p_{21}, p_{22}) = 1$. Denote by $\langle p_{21}, p_{22}\rangle$ the ideal generated by the polynomials p_{21} and p_{22}. Since

$$\langle p_{21}, p_{22}\rangle = \langle \mathsf{gcd}(p_{21}, p_{22})\rangle = \langle 1\rangle = \mathbb{R}[\zeta],$$

there are polynomials p_{11} and p_{12} such that $\det(P_1) = 1$ where

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \tag{6.4}$$

Then by construction we have

$$PC = \begin{pmatrix} * & * & \dots & * \\ 0 & * & \dots & * \end{pmatrix}$$

where $*$ denotes some element.

Now consider the following matrix:

$$\tilde{P} = \begin{pmatrix} I_{s_1} & 0 & 0 & 0 & 0 \\ 0 & p_{11} & 0 & p_{12} & 0 \\ 0 & 0 & I_{s_2} & 0 & 0 \\ 0 & p_{21} & 0 & p_{22} & 0 \\ 0 & 0 & 0 & 0 & I_{s_3} \end{pmatrix} \tag{6.5}$$

where I_i is identity $i \times i$ matrix and $k = s_1 + s_2 + s_3 + 2$ (if $s_i = 0$ then we delete the corresponding row and column). Now if we construct polynomials p_{ij} like above then $\det(\tilde{P}) = 1$ and multiplying A by $\tilde{P}_1$ produces a zero in the row $s_1 + s_2 + 2$.

Then choosing appropriate matrices of this form we can form a product $T = \tilde{P}_j \dots \tilde{P}_1$ which has the properties stated in the theorem. ☐

Let us define $m \times m$ matrix $\tilde{\mathsf{U}}$ by $\tilde{\mathsf{U}}_{ij} = \mathsf{U}_{ij}$ except that $\tilde{\mathsf{U}}_{mm} = \mathsf{gcd}(\mathsf{U}_{mm}, \dots, \mathsf{U}_{km})$.

Substituting now D_n for ζ, we consider the upper triangular square system of ordinary differential equations

$$\tilde{\mathsf{U}}(\xi', D_n)u(x_n) = 0, \quad x_n > 0. \tag{6.6}$$

LEMMA 6.1. *For any fixed $\xi' \neq 0$, the solution spaces of the systems* (6.2) *and* (6.3) *are equal. In particular, for any fixed $\xi' \neq 0$*

$$\dim\left(\ker(\mathsf{A}(x,\xi',D_n)) \cap \mathfrak{M}_+\right) = \dim\left(\ker(\tilde{\mathsf{U}}(x,\xi',D_n)) \cap \mathfrak{M}_+\right) = \nu.$$

Proof. The statement of this lemma follows from equivalence of systems (6.6) and (6.2), and Lemma 4.1. □

6.3. *Algebraic criterion for checking the SL-condition*

First we fix some $\xi' \neq 0$. Let $\zeta_1, \dots, \zeta_\nu$ be the roots of the polynomial $p_{\mathsf{A}} \in \mathbb{R}[\zeta]$ lying in the upper half of the complex plane. We set

$$p_{\mathsf{A}}^+ = (\zeta - \zeta_1)\cdots(\zeta-\zeta_\nu) = \sum_{j=0}^{\nu} b_j \zeta^j.$$

and introduce the polynomials p_l^+, $l = 1, \dots, \nu$, as in (2.2).

LEMMA 6.2. *Let A_s be some $m \times m$ submatrix of the matrix A. Then the columns of the matrix*

$$W_l(x_n) = \frac{1}{2\pi i}\oint_{\gamma_+} \frac{\mathsf{adj}(\mathsf{A}_s(\zeta)) p_l^+(\zeta) e^{i\zeta x_n}}{p_{\mathsf{A}}^+(\zeta)}\, d\zeta, \quad l = 1,\dots,\nu,$$

belong to the space $\ker(\mathsf{A}(D_n)) \cap \mathfrak{M}_+$.

Proof. Note that

$$\mathsf{A}(D_n) W_l(x_n) = \frac{1}{2\pi i}\oint_{\gamma_+} \frac{\mathsf{A}(\zeta)\,\mathsf{adj}(\mathsf{A}_s(\zeta)) p_l^+(\zeta) e^{i\zeta x_n}}{p_{\mathsf{A}}^+(\zeta)}\, d\zeta, \quad l = 1,\dots,\nu.$$

Hence, if we show that all elements of the matrix $\tilde{\mathsf{A}} = \mathsf{A}\,\mathsf{adj}(\mathsf{A}_s)$ are divisible by the polynomial p_{A}^+, then by the Cauchy integral theorem we get $\mathsf{A}(D_n)W_l(x_n) = 0$. Since $\Re(i\zeta) < 0$, we have that the columns of W_l belong to $\ker(\mathsf{A}(D_n)) \cap \mathfrak{M}_+$.

Let us now show that elements of $\tilde{\mathsf{A}}$ are divisible by p_{A}^+. Let us denote by $(a_{i1}, \dots, a_{im})$ the ith row of the matrix A. Then by the definition of the adjoint matrix for the ith row of $\tilde{\mathsf{A}}$, we get

$$\tilde{\mathsf{A}}_{ij} = \det(\mathsf{A}_s),\ \tilde{\mathsf{A}}_{ij} = 0,\ j \neq j,$$

if the ith row of the matrix A belongs to A_s, or

$$\tilde{\mathsf{A}}_i = (a_{i1}, \dots, a_{im})\,\mathsf{adj}(\mathsf{A}_s),$$

otherwise. In case of the first possibility by the definition of the polynomial p_{A}^+ every $\tilde{\mathsf{A}}_{ij}$ is divisible by p_{A}^+.

Let us now consider the second case. By the definition of the adjoint matrix, the elements of $\mathsf{adj}(\mathsf{A}_s)$ are $\big(\mathsf{adj}(\mathsf{A}_s)\big)_{ij} = (-1)^{i+j} B_{ji}$ where B_{ij} is the (i,j) minor of the matrix A_s. Hence,

$$\tilde{\mathsf{A}}_{ij} = a_{i_1}(-1)^{1+j} B_{j1} + \dots + a_{i_m}(-1)^{m+j} B_{jm}.$$

But then $\tilde{\mathsf{A}}_{ij}$ is $\pm$ the determinant of the $m \times m$ submatrix of A which consists of the row $(a_{i1}, \dots, a_{im})$ and all rows of A_s except the jth row. So again by the definition of the polynomial p_{A}^+ every $\tilde{\mathsf{A}}_{ij}$ is divisible by p_{A}^+. □

LEMMA 6.3. *There is a $m \times m$ submatrix of A such that some element of its adjoint is not divisible by p_{A}^+.*

Proof. Suppose the contrary, i.e. all elements of all submatrices $\mathsf{adj}(\mathsf{A}_s)$ are divisible by p_{A}^+: $\mathsf{adj}(\mathsf{A}_s) = p_{\mathsf{A}}^+ C_s$ for all $s = 1, \dots, r$. Setting $q_s = \det(\mathsf{A}_s)$ and by the property of adjoint matrix we get $\det(\mathsf{adj}(\mathsf{A}_s)) = q_s^{m-1}$. So we have $(p_{\mathsf{A}}^+)^m \det(C_s) = q_s^{m-1}$. Since $p_{\mathsf{A}} = \mathsf{gcd}(q_1, \dots, q_r)$, we get $q_s = p_{\mathsf{A}}^+ \tilde{q}_s$ for all $s = 1, \dots, r$. Thus, $p_{\mathsf{A}}^+ \det(C_s) = (\tilde{q}_s)^{m-1}$. Hence, $\tilde{q}_s$, $s = 1, \dots, r$, are divisible by p_{A}^+ and so p_{A} is divisible by $(p_{\mathsf{A}}^+)^2$. But this contradicts the definition of p_{A}^+.

□

Let A_s be as in Lemma 6.3 and denote by v some column of the matrix $\mathsf{adj}(\mathsf{A}_s)$ which is nonzero modulo p_{A}^+. And as in the square case we set

$$\omega^l(x_n) = \frac{1}{2\pi i}\oint_{\gamma_+} \frac{v(\zeta)p_l^+(\zeta)e^{i\zeta x_n}}{p_{\mathsf{A}}^+(\zeta)}\,d\zeta, \quad l = 1,\dots,\nu.$$

and let ω be a matrix with columns ω^l.

LEMMA 6.4. *The columns of ω are a basis of the space* $\ker(\mathsf{A}(D_n)) \cap \mathfrak{M}_+$.

Proof. Due to Lemma 6.2 and Lemma 6.3, the proof is the same as in square case (see Lemma 4.3). □

Let us now consider the vector $h = \mathsf{B}_w v$. Dividing each element of h by the polynomial p_{A}^+ we get

$$h = q\,p_{\mathsf{A}}^+ + \mathsf{h} \qquad \text{where} \qquad \mathsf{h} = \sum_{\tau=0}^{\nu-1} \mathsf{h}^\tau \zeta^\tau$$

Let us introduce the $\tilde{\nu} \times \nu$ matrix $\mathcal{H}$

$$\mathcal{H} = \big(\mathsf{h}^0,\dots,\mathsf{h}^{\nu-1}\big). \tag{6.7}$$

Now we formulate the algebraic criterion for checking the coerciveness condition.

THEOREM 6.3. *Following statements are equivalent*

(i) *an operator (A, B) satisfies the SL-condition;*

(ii) *for any $x \in \Gamma$ and $\xi' \neq 0$, there is a column v of some matrix $\mathsf{adj}(\mathsf{A}_s)$ that is not divisible by p_{A}^+ such that* $\operatorname{rank}(\mathcal{H}, d) = \nu$ *for all* $d \in \ker((\Phi_1^{22})_w)$ *and* $\dim\big(\ker((\Phi_1^{22})_w)\big) = \nu$ *where $\mathcal{H}$ is defined as in* (6.7) *using v.*

Proof. *(ii) implies (i).* First we fix some $x \in \Gamma$ and $\xi' \neq 0$. We take a column v of some matrix $\mathsf{adj}(\mathsf{A}_s)$ that is not divisible by p_{A}^+ such that $\operatorname{rank}(\mathcal{H}, d) = \nu$ for all $d \in \ker((\Phi_1^{22})_w)$. Since $(\Phi_1^{22})_w$ is a linear operator, we get $0 \in \ker((\Phi_1^{22})_w)$. This implies that $\operatorname{rank}(\mathcal{H}) = \nu$. So for all $d \in \ker((\Phi_1^{22})_w)$ we have $\operatorname{rank}(\mathcal{H}) = \operatorname{rank}(\mathcal{H}, d)$ and therefore the system $\mathcal{H}c = d$ has a solution. Using this vector c we construct the following function $u(x_n) = \omega(x_n)c$. Lemma 6.4 implies that $u \in \ker(\mathsf{A}(D_n)) \cap \mathfrak{M}_+$. By Lemma 4.4 we get $\mathsf{B}_w(D_n)u(x_n)|_{x_n=0} = \mathcal{H}c = d$. So we have $\ker((\Phi_1^{22})_w) \subset \operatorname{im}(\mathsf{B}_w)$. The definition of compatibility operator implies that $\operatorname{im}(\mathsf{B}_w) \subset \ker((\Phi_1^{22})_w)$. Hence, $\ker((\Phi_1^{22})_w) = \operatorname{im}(\mathsf{B}_w)$. So $\dim\big(\operatorname{im}(\mathsf{B}_w)\big) = \nu$. By Lemma 6.1 we have that $\dim\big(\ker(\mathsf{A}(D_n)) \cap \mathfrak{M}_+\big) = \nu$. Since B_w is a linear operator, we get $\dim\big(\ker(\mathsf{B}_w)\big) = 0$. So $\ker(\mathsf{B}_w) = \{0\}$ and hence, the complex (6.1) is exact.

(i) implies (ii). Suppose now that the complex (6.1) is exact. Take some $x \in \Gamma$ and $\xi' \neq 0$. This together with linearity of the operator B_w imply that $\dim\big(\operatorname{im}(\mathsf{B}_w)\big) = \dim\big(\ker((\Phi_1^{22})_w)\big) = \nu$. Take some column v of some $\mathsf{adj}(\mathsf{A}_s)$ that is not divisible by p_{A}^+ and construct the matrix ω using v. According to Lemma 6.4 the columns of the matrix ω are a basis of space $\ker(\mathsf{A}(D_n)) \cap \mathfrak{M}_+$. Since B_w is a linear operator, $\ker(\mathsf{B}_w) = \{0\}$ and $\dim\big(\operatorname{im}(\mathsf{B}_w)\big) = \nu$, we get that the columns of $\mathsf{B}_w\omega(x_n)|_{x_n=0} = \mathcal{H}$ are a basis of the space $\operatorname{im}(\mathsf{B}_w)$. We have also that $\ker((\Phi_1^{22})_w) = \operatorname{im}(\mathsf{B}_w)$ and hence, $\operatorname{rank}(\mathcal{H}, d) = \nu$ for all $d \in \ker((\Phi_1^{22})_w)$. □

REMARK 6.2. *If condition (ii) of Theorem 6.3 holds, then for any column of any matrix $\mathsf{adj}(\mathsf{A}_1),\dots,\mathsf{adj}(\mathsf{A}_r)$ that is not divisible by p_{A}^+,* $\operatorname{rank}(\mathcal{H}, d) = \nu$ *for all* $d \in \ker((\Phi_1^{22})_w)$ *where $\mathcal{H}$ is defined using this column.*

6.4. *Computational test for checking the SL-condition in case of two independent variables*

Let $\hat{p}_{\mathsf{A}} \in \mathbb{K}[\zeta]$, where $\mathbb{K}$ is as before, be the dehomogenised characteristic polynomial and let us denote by $\hat{\mathsf{A}}$, $\hat{\mathsf{B}}_w$, $(\hat{\Phi}_1^{22})_w$ the dehomogenised symbol matrices. Also we denote the splitting field of $\hat{p}_{\mathsf{A}}$ by $\mathbb{K}(\alpha)$. It is evident that $\hat{p}_{\mathsf{A}} = \mathsf{gcd}(\det(\hat{\mathsf{A}}_1),\dots,\det(\hat{\mathsf{A}}_r))$ and we have homomorphisms ι and $\tilde{\iota}$ as in (4.7) and (4.8).

We define polynomials $\hat{p}_{\mathsf{A}}^+$, $\tilde{p}_{\mathsf{A}}^+$ and $\tilde{p}_{\mathsf{A}}^-$ as in (4.9) and (4.10). As in square case we will work with polynomial $\hat{p}_{\mathsf{A}}^+$ in our computational test. The following result is similar to Lemma 4.5.

LEMMA 6.5. *There is an element $\hat{d}$ of some $\mathsf{adj}(\hat{\mathsf{A}}_s)$ such that*

1. *$\hat{d}$ is not divisible by $\hat{p}_{\mathsf{A}}^+$;*

2. *$\hat{d}$ is not divisible by $\tilde{\iota}(\hat{p}_{\mathsf{A}}^{+})$;*
3. *the corresponding element d of $\mathsf{adj}(\mathsf{A}_s)$ is not divisible by $\tilde{p}_{\mathsf{A}}^{+}$;*
4. *for any fixed $\xi_1 \neq 0$ the corresponding element d of $\mathsf{adj}(\mathsf{A}_s)$ is not divisible by p_{A}^{+}.*

Suppose that $\hat{d}$ is in the j'th column of $\mathsf{adj}(\hat{\mathsf{A}}_s)$. We denote by $\hat{v}$ (resp. v) the j'th column of $\mathsf{adj}(\hat{\mathsf{A}}_s)$ (resp. $\mathsf{adj}(\mathsf{A}_s)$).

Now for any fixed $\xi_1 \neq 0$, we construct the matrix $\mathcal{H}$ as in (6.7) using the j'th column of $\mathsf{adj}(\mathsf{A}_s)$ and polynomial p_{A}^{+}. Then working with polynomials in two variables in the same way we construct a matrix $\tilde{\mathcal{H}}$ using the same j'th column of $\mathsf{adj}(\mathsf{A}_s)$ and polynomial $\tilde{p}_{\mathsf{A}}^{+}$. It is immediate that for any fixed $\xi_1 \neq 0$

$$\mathcal{H} = \begin{cases} \tilde{\mathcal{H}} & , \text{ if } \xi_1 > 0, \\ \overline{\tilde{\mathcal{H}}} & , \text{ if } \xi_1 < 0. \end{cases} \tag{6.8}$$

Let us now consider dehomogenised symbols and construct a matrix $\hat{\mathcal{H}}$ using the j'th column of $\mathsf{adj}(\hat{\mathsf{A}}_s)$ and polynomial $\hat{p}_{\mathsf{A}}^{+}$. Is is easily seen that the matrix constructed using the j'th column of $\mathsf{adj}(\hat{\mathsf{A}}_s)$ and polynomial $\tilde{\iota}(\hat{p}_{\mathsf{A}}^{+})$ is $\iota(\hat{\mathcal{H}})$. Moreover one can check that

$$\tilde{\mathsf{h}}_i^\tau = \xi_1^{r_i+(m-1)q-\tau}\iota(\hat{\mathsf{h}}_i^\tau), \quad \tau = 0,\dots,\nu-1, \quad i = 1,\dots,\tilde{\nu}. \tag{6.9}$$

Now we are ready to formulate the computational test for the SL-condition

THEOREM 6.4. *An operator (A, B) satisfies the SL-condition if and only if*

(i) $\operatorname{rank}((\hat{\Phi}_1^{22})_w) = \tilde{\nu} - \nu$;

(ii) $\operatorname{rank}(\hat{\mathcal{H}}) = \operatorname{rank}(\hat{\mathcal{H}}, \hat{k}^l) = \nu$, $l = 1,\dots,\nu$, *where* $\hat{k}^1,\dots,\hat{k}^\nu$ *is a basis of the vector space* $\ker((\hat{\Phi}_1^{22})_w)$.

Note that elements of $(\hat{\Phi}_1^{22})_w$ are in $\mathbb{K}$ because Φ_1^{22} is an operator on the boundary only. Hence $\ker((\hat{\Phi}_1^{22})_w)$ is really a vector space and not just a module.

Proof. ¿From Theorem 6.3, we deduce that the SL-condition is equivalent to the following: for all $\xi_1 \neq 0$

(1) $\dim(\ker((\Phi_1^{22})_w)) = \nu$ and

(2) $\operatorname{rank}(\mathcal{H}, d) = \nu$ for all $d \in \ker((\Phi_1^{22})_w)$.

First note that condition (1) is equivalent to: for all $\xi_1 \neq 0$, $\operatorname{rank}((\Phi_1^{22})_w) = \tilde{\nu} - \nu$. Then for a fixed $\xi_1 \neq 0$, the matrix $(\Phi_1^{22})_w$ is obtained from the matrix $(\hat{\Phi}_1^{22})_w$ by multiplication of each j'th column by $\xi_1^{r_j}$ and each i'th row by $\xi_1^{\delta_i}$. Since the rank does not change under multiplication of columns and rows by nonzero elements, condition (1) is also equivalent to condition (i).

Then fixing some $\xi_1 \neq 0$ we see that $d \in \ker((\Phi_1^{22})_w)$ has the following form:

$$d = \left(\hat{d}_1 \xi_1^{\tilde{m}+r_1}, \dots, \hat{d}_{\tilde{\nu}} \xi_1^{\tilde{m}+r_{\tilde{\nu}}}\right) \tag{6.10}$$

where $\hat{d} = \left(\hat{d}_1,\dots,\hat{d}_{\tilde{\nu}}\right) \in \ker((\hat{\Phi}_1^{22})_w)$ and $\tilde{m} \in \mathbb{Z}$. Conversely, if $\hat{d} \in \ker((\hat{\Phi}_1^{22})_w)$, then $d \in \ker((\Phi_1^{22})_w)$ for any $\tilde{m}$ and $\xi_1 \neq 0$.

Now (6.8), (6.10) and (6.9) imply that the condition (2) is equivalent to the following statements:

(2)' for all $\xi_1 \neq 0$, $\operatorname{rank}(\tilde{\mathcal{H}}, d) = \nu$ for any $d \in \ker((\Phi_1^{22})_w)$

(2)" $\operatorname{rank}(\iota(\hat{\mathcal{H}}), \hat{d}) = \nu$ for any $\hat{d} \in \ker((\hat{\Phi}_1^{22})_w)$.

Equalities $\hat{d} = \iota(\hat{d})$, $\det(\iota(C)) = \iota(\det(C))$ for any matrix C, and injectivity of ι yield that the condition (2)" is equivalent to $\operatorname{rank}(\hat{\mathcal{H}}, \hat{d}) = \nu$ for any $\hat{d} \in \ker((\hat{\Phi}_1^{22})_w)$. But by elementary algebra this condition is equivalent to $\operatorname{rank}(\hat{\mathcal{H}}) = \operatorname{rank}(\hat{\mathcal{H}}, \hat{k}^l) = \nu$, $l = 1,\dots,\nu$, where $\hat{k}^1,\dots,\hat{k}^\nu$ is a basis of $\ker((\hat{\Phi}_1^{22})_w)$. □

COROLLARY 6.1. *Let $\nu = \tilde{\nu}$. An operator (A, B) satisfies the SL-condition if and only if rank$(\hat{\mathcal{H}}) = \nu$.*

EXAMPLE 6.1. *Consider the transformation of the two-dimensional Laplace equation $u_{20} + u_{02} = 0$ to the first order elliptic system*

$$A: \quad \begin{cases} y_{10}^1 + y_{01}^2 = 0, \\ y_{01}^1 - y_{10}^2 = 0, \\ y_{10}^3 - y^1 = 0, \\ y_{01}^3 - y^2 = 0, \end{cases} \qquad \textit{in} \quad \mathbb{R}_+^2 = \{x \in \mathbb{R}^2 : x_2 > 0\}, \tag{6.11}$$

with the following boundary conditions

$$B: \quad \begin{cases} y^1 = 0, \\ y^3 = 0, \end{cases} \qquad \text{on} \quad \partial\mathbb{R}^2_+. \tag{6.12}$$

Note that these boundary conditions are the Dirichlet condition ($y^3 = 0$ on $\partial\mathbb{R}^2_+$) for the Laplace equation and a differential consequence of the relations $y^3 = 0$ and $y^3_{10} - y^1 = 0$ on $\partial\mathbb{R}^2_+$ (the tangent part of the operator A). We will prove that these boundary conditions satisfy the SL-condition.

Now we will construct the component Φ_1^{22} of a compatibility operator Φ_1 for the boundary value problem operator (A, B). Note that the operator (A, B) is normalised and the tangent part of the operator A is $A^\tau y = y^3_{10} - y^1$. Let us define a differential operator

$$\Phi^\tau y = (A^\tau y, By) = (y^3_{10} - y^1, y^1, y^3)\,.$$

To construct a compatibility operator Φ_1^τ for the operator Φ^τ we compute the syzygy module for the module generated by the rows of the matrix

$$\begin{pmatrix} -1 & 0 & \xi_1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using Singular *we get $(-1, -1, \xi_1)$ and hence,*

$$\Upsilon^\tau(f^\tau, g^1, g^2) = -f^\tau - g^1 + \frac{\partial g^2}{\partial x_1}.$$

So the component Φ_1^{22} of the compatibility operator Φ_1 and its weighted principal symbol with weights $r_1 = r_2 = 0$ and $\delta_1 = 1$ are

$$\Phi_1^{22}(g^1, g^2) = -g^1 + \frac{\partial g^2}{\partial x_1}, \quad (\mathsf{\Phi}_1^{22})_w = \begin{pmatrix} 0 & \xi_1 \end{pmatrix}, \quad (\hat{\mathsf{\Phi}}_1^{22})_w = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Let us consider the principal symbol of the operator A

$$\mathsf{A} = \begin{pmatrix} \xi_1 & \zeta & 0 \\ \zeta & -\xi_1 & 0 \\ 0 & 0 & \xi_1 \\ 0 & 0 & \zeta \end{pmatrix}.$$

So the characteristic polynomial of the operator A is

$$p_{\mathsf{A}} = \mathsf{gcd}\big(-\zeta^3 - \zeta\xi_1^2, -\zeta^2\xi_1 - \xi_1^3\big) = \zeta^2 + \xi_1^2, \quad \hat{p}_{\mathsf{A}} = \zeta^2 + 1, \quad \hat{p}^+_{\mathsf{A}} = \zeta - i.$$

Thus $\operatorname{rank}((\hat{\mathsf{\Phi}}_1^{22})_w) = \tilde{\nu} - \nu = 1$. *Hence condition (i) of Theorem 6.4 holds. Let us then consider the following submatrix*

$$\hat{\mathsf{A}}_{123} = \begin{pmatrix} 1 & \zeta & 0 \\ \zeta & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

of the dehomogenised principal symbol $\hat{\mathsf{A}}$. Computing with Singular *the adjoint of $\hat{\mathsf{A}}_{123}$ we get*

$$\mathsf{adj}(\hat{\mathsf{A}}_{123}) = \begin{pmatrix} -1 & -\zeta & 0 \\ -\zeta & 1 & 0 \\ 0 & 0 & -\zeta^2 - 1 \end{pmatrix}$$

Let $\hat{v}$ be the first column of $\mathsf{adj}(\hat{\mathsf{A}}_{123})$. The vector $\hat{v}$ is nonzero modulo $\hat{p}^+_{\mathsf{A}}$, since the reduction of $\hat{v}$ with respect to $\hat{p}^+_{\mathsf{A}}$ is $(-1, -i, 0)$. Note that

$$\hat{\mathsf{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So we get $\hat{h} = \hat{\mathsf{B}}\hat{v} = (-1, 0)$. Reducing the elements of $\hat{h}$ with respect to $\hat{p}^+_{\mathsf{A}}$ we get $\hat{\mathsf{h}} = (-1, 0)$. Since $\nu = 1$, we have $\hat{\mathcal{H}} = \hat{\mathsf{h}}$. So $\operatorname{rank}(\hat{\mathcal{H}}) = \operatorname{rank}(\hat{\mathcal{H}}, \hat{k}) = 1$ where $\hat{k} = (1, 0)$ is a basis of the vector space $\ker((\hat{\mathsf{\Phi}}_1^{22})_w)$. Hence condition (ii) of Theorem 6.4 holds and so the boundary conditions (6.12) *satisfy the SL-condition.*

EXAMPLE 6.2. *Let us consider the boundary value problem* (A_0, B) *defined by* (5.13) *and* (5.14) *where* A_0 *is the involutive form of the Stokes system.*

The dehomogenised principal symbol of A_0 *is*

$$\hat{\mathsf{A}}_0 = \begin{pmatrix} -\zeta^2 - 1 & 0 & 0 \\ 0 & -\zeta^2 - 1 & 0 \\ 0 & 0 & -\zeta^2 - 1 \\ 0 & 0 & 0 \\ 1 & \zeta & 0 \\ \zeta & \zeta^2 & 0 \end{pmatrix}.$$

Hence $\hat{p}_{\mathsf{A}} = (\zeta^2 + 1)^2$ *and* $\nu = 2$. *The part* Φ_1^{22} *of the compatibility operator for* (A_0, B) *was computed in* (5.15)*. The dehomogenised weighted symbol of* Φ_1^{22} *is*

$$(\hat{\Phi}_1^{22})_w = \begin{pmatrix} 1 & -2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

So $\operatorname{rank}((\hat{\Phi}_1^{22})_w) = 2 \neq \tilde{\nu} - \nu = 3$. *Hence condition (i) of Theorem 6.4 does not hold and* (A_0, B) *does not satisfy the SL-condition.*

7. *Conclusions and perspectives*

We have shown above how to check constructively the validity of the SL–condition in the 2 variable case. This case is already important in PDE computations; however, it would evidently be desirable to extend our results to the case of arbitrary number of variables.

When there are “too many” boundary conditions we need the compatibility operator to perform the test. One may wonder why should one try to impose more boundary conditions than strictly necessary. On the other hand one could ask the same question about PDE systems; yet it is clear that analysing just square systems is not enough. Perhaps the same will happen with boundary conditions: when the systems get more complicated there may arise situations where it is “natural” or important to consider “too many” boundary conditions. However, as far as we know there has been no work in this direction in the numerical analysis of PDEs, so it is at present not clear how important this will be in the future.

In formulating our results we have supposed that the system is given in a specific (local) coordinate system. Obviously in practice we do not want to make explicit coordinate transformations, so a natural goal would be to formulate the test in original coordinate system. On a more technical level it is not clear if it is really necessary to factor the characteristic polynomial. Explicit factorisation of even quite simple polynomials can be very time consuming. Moreover, if we can avoid factoring in the 2 variable case, perhaps this alternative approach could be generalised to many variable case. Finally a natural way to extend our work is to apply this approach also to overdetermined parabolic and hyperbolic systems. We hope to address all these issues in future papers.

8. *Appendix*

Useful commands in computer algebra systems Singular *and* MuPad

As we have seen, symbols of differential operators are naturally viewed as modules generated by the rows of the symbol matrix. However, in Singular (and in commutative algebra textbooks in general) modules are generated by the columns of matrices. Hence in the following commands we will often need to transpose matrices. To use some of the commands in Singular one needs to load some appropriate libraries. To access all libraries one can use the command:

$>$ LIB ”all.lib”;

8.1. *Finding the splitting field of the characteristic polynomial and its roots*

$>$ ring $r = 0, y, dp$;

$>$ poly $p = y^6 + y^4 + y^2 + 1;$
$>$ def $r1$ = Roots$(p);$
$>$ setring $r1;$
$>$ roots;
$a^3, -a^3, a, -a, -a^2, a^2$
$>$ minpoly;
$a^4 + 1$

8.2. *Determine the admissible combinations of the roots of the characteristic polynomial*
$>$ ring $r2 = complex, a, dp;$
$>$ poly $pmin = a^4 + 1;$
$>$ list l = solve$(pmin);$
$>$ poly $root1 = a^3;$
$>$ substitute$(root1, a, l[1]);$
$-0.70710678 + i\ 0.70710678$

8.3. *Computations in the SL–test*
$>$ ring $r = (0, a), z, dp;$ minpoly $= a^2 + 1;$
$>$ poly $p = z - a;$
$>$ matrix $ma[3][3] = 1, z, 0, z, -1, 0, 0, 0, 1;$
$>$ matrix $mb[2][3] = 1, 0, 0, 0, 0, 1;$

// Finding of a vector v
$>$ matrix $maadj$ = adjoint$(ma);$
$>$ ideal $id = p;$ ideal $idstd$ = std$(id);$
$>$ int m = ncols$(maadj);$
int $i;$ int$j;$
for $(j = 1; j <= m; j = j + 1)$
{
for $(i = 1; i <= m; i = i + 1)$
{
$maadj[i, j]$ = reduce$(maadj[i, j], idstd);$
}
}
// After the reduction we notice that the first column is nonzero.
$>$ matrix $v[3][1] = -1, -z, 0;$

// Definition of a vector h and construction of a vector h
$>$ matrix $mh = mb * v;$
$>$ int m = ncols$(mh);$
int n = nrows$(mh);$
int $i;$ int $j;$
for $(j = 1; j <= m; j = j + 1)$
{
for $(i = 1; i <= n; i = i + 1)$
{
$mh[i, j]$ = reduce$(mh[i, j], idstd);$
}
}

// Construction of a matrix $\mathcal{H}$
$>$ matrix mco = coeffs$(mh, z);$
int d = nrows$(mco)/n;$
int $i;$ int $j;$ int $i1;$ int $k;$ int $e1;$ int $e2;$
matrix $mco1[n][d * m];$

```
for (i = 1; i <= n; i = i + 1)
{
k = 0;
e1 = (i − 1) ∗ d + 1;
e2 = i ∗ d;
for (i1 = e1; i1 <= e2; i1 = i1 + 1)
{
for (j = 1; j <= m; j = j + 1)
{
k = k + 1;
mco1[i, k] = mco[i1, j];
}
}
}
```

8.4. *Computation of the compatibility operator when there is no boundary operator*

```
> ring r = 0, (z, x), dp;
> matrix m[3][3] = −1, 0, x, 1, 0, 0, 0, 0, 1;
> matrix s = transpose(syz(transpose(m)));
```

8.5. *Computation of the tangent part of the operator A*

```
> ring r = 0, (z, x), (lp, c);
// Here we define TOP ordering with z > x.
> module m = [0, 0, 0, −x, 0, 1, −z, 0], [0, 0, 0, z, −x, 0, 0, 1],
[0, 0, 0, 0, 0, −x, 0, −z], [x, 0, 0, −1, 0, 0, 0, 0], [z, 0, 0, 0, 0, 0, −1, 0],
[0, x, 0, 0, −1, 0, 0, 0], [0, z, 0, 1, 0, 0, 0, 0], [0, 0, x, 0, 0, −1, 0, 0],
[0, 0, z, 0, 0, 0, 0, −1], [0, 0, 0, z, 0, 0, −x, 0], [0, 0, 0, x, z, 0, 0, 0],
[0, 0, 0, 0, 0, z, 0, −x];
> module ms = transpose(std(m));
> print(ms);
0, 0, 0, 0, x, 0, −x, −1,
0, 0, x, 0, 0, −1, 0, 0,
0, x, 0, 0, −1, 0, 0, 0,
x, 0, 0, −1, 0, 0, 0, 0,
0, 0, 0, 0, 0, x, 0, z,
0, 0, 0, x, 0, −1, z, 0,
0, 0, 0, 0, 0, z, 0, −x,
0, 0, 0, x, z, 0, 0, 0,
0, 0, 0, z, 0, 0, −x, 0,
0, 0, z, 0, 0, 0, 0, −1,
0, z, 0, 1, 0, 0, 0, 0,
z, 0, 0, 0, 0, 0, −1, 0
```

8.6. *Completion of a system to involutive form with MuPad*

```
> LDF := Dom::LinearDifferentialFunction(Vars = [[x1, x2, x3], [y]], Rest = [Types = "Indep"]) :
> sys := map([y([x3, x3]) − x2 ∗ y([x1, x1]), y([x2, x2])], LDF) :
> detools :: complete(sys, Output = 3);
```

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