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Compatibility Complexes for Overdetermined Boundary Problems

Katsiaryna Krupchyk and Jukka Tuomela

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1 Introduction

When analysing general systems of partial differential equations (PDEs) it is important to check if the system is involutive, and if not then transform it to the involutive form. For example, in [6] we proved that some systems may not be elliptic initially (even in the general sense), but their involutive forms are elliptic. The technical definition of the involutive form is quite complicated (see [10], [13] and [12] and for the actual definition). However, essentially the involutivity means that one has to find all integrability conditions (or differential consequences) of the given system up to some order.

Now the involutive form is in general overdetermined. To study overdetermined systems one needs to find all solvability conditions, or more generally, to construct a

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compatibility complex for the corresponding overdetermined operator. We show that for linear partial differential operators with constant coefficients one can compute the compatibility complex by simply computing the free resolution of the module generated by the rows of operators. Incidentally this shows that the length of the compatibility complex is at most the number of independent variables.

However, to study boundary value problems one needs to compute the compatibility operators involving the boundary operators. To perform this task it is convenient to further transform the involutive system to a normalised system. Roughly speaking, an operator is normalised if it is a first order involutive operator and there are no (explicit or implicit) algebraic (i.e., non-differential) relations between dependent variables. A boundary value problem operator is normalised if the system is normalised and the boundary conditions contain only differentiation in directions tangent to the boundary. Computing the compatibility complex for a normalised boundary problem operator is not as straightforward as the simple free resolution, but anyway we show that the problem can be formulated again with modules, and choosing suitable module orderings we can compute the necessary information by Gröbner basis techniques. We explain how to construct the compatibility complex for a general boundary problem operator using the compatibility complex for a corresponding normalised boundary problem operator.

The construction of compatibility complexes is useful and even necessary when investigating the well-posedness of overdetermined boundary value problems. In [8] and [7] we have used compatibility complexes to study well-posedness of elliptic problems and moreover, in [9] compatibility complexes are even used in the numerical solution of PDEs. Note that constructions given in this paper are also essential in the theory of overdetermined parabolic and hyperbolic systems of PDEs.

2 Preliminaries

2.1 Formal Theory of PDEs

Let us consider a smooth[1] manifold $\mathcal{X}$. Let $\pi : V \to \mathcal{X}$ be a vector bundle over $\mathcal{X}$ and let $\pi^q : J_q(V) \to \mathcal{X}$ be the bundle of q-jets of the bundle V. Let us also introduce the canonical projections

$$\pi^q_r : J_q(V) \to J_r(V),$$

for $r < q$. Let y be a section of the bundle V. Then its qth prolongation, a section of $J_q(V)$, is denoted by $j^q y$. We write $C^\infty(V)$ for the space of smooth sections of the bundle V.

Definition 2.1 A (partial) differential equation of order q on V is a subbundle $\mathcal{R}_q$ of $J_q(V)$. Solutions of $\mathcal{R}_q$ are its (local) sections.

[1] In this paper smooth means infinitely differentiable.

We will only consider linear problems, so $\mathcal{R}_q$ will be a vector bundle. Suppose V^0 and V^1 are two (vector) bundles. A linear qth order differential operator A can be thought of as a linear map $C^\infty(V^0) \to C^\infty(V^1)$. Then we can associate to A a bundle map $\varphi_A : J_q(V^0) \to V^1$ by the formula $A = \varphi_A j^q$. Now with φ_A one can represent a differential equation as a zero set of a bundle map, $\mathcal{R}_q = \ker \varphi_A$, or $\varphi_A(x, j^q y(x)) = 0$.

Definition 2.2 The differential operator $j^r A : C^\infty(V^0) \to C^\infty(J_r(V^1))$ is said to be the rth prolongation of A. The associated bundle map is denoted by $p_r(\varphi_A)$.

Then we can define the rth prolongation of $\mathcal{R}_q$ by $\mathcal{R}_{q+r} = \ker p_r(\varphi_A)$. We also define $\mathcal{R}^{(s)}_{q+r} = \pi^{q+r+s}_{q+r}(\mathcal{R}_{q+r+s})$ for all $s \geq 0$. Note that $\mathcal{R}^{(s)}_{q+r} \subset \mathcal{R}_{q+r}$, but in general these sets are not equal.

Definition 2.3 A differential operator A is sufficiently regular if $\mathcal{R}^{(s)}_{q+r}$ is a vector bundle for all $r \geq 0$ and $s \geq 0$.

If $\mathcal{X} \subset \mathbb{R}^n$ and the operator A has constant coefficients, then A is sufficiently regular.

Definition 2.4 A differential operator A of order q is formally integrable if A is sufficiently regular and $\mathcal{R}^{(1)}_{q+r} = \mathcal{R}_{q+r}$ for all $r \geq 0$.

The formal integrability of an operator A of order q means that for any $r \geq 1$, all the differential consequences of order $q + r$ of the relations $Ay = 0$ may be obtained by means of differentiations of order no greater than r, and application of linear algebra.
The formal integrability cannot in general be checked in practice because there is an infinite number of conditions. Hence we need a stronger property, the *involutivity* of the system, which implies formal integrability, and can be checked in a finite number of steps. For the actual definition of involutivity we refer to [12], [10], [11]. There is the following important result.

Theorem 2.5 *For a given sufficiently regular system $\mathcal{R}_q$ there are numbers r and s such that $\mathcal{R}^{(s)}_{q+r}$ is involutive,*

In practice to complete a system to the involutive form one may use DETOOLS package [2] in the computer algebra system MUPAD [4].
In the context of the formal theory the principal symbol of the system is defined as follows. Let us first define the embedding ε_q by requiring that the following complex be exact

$$0 \longrightarrow S^q(T^*\mathcal{X}) \otimes V \xrightarrow{\varepsilon_q} J_q(V) \xrightarrow{\pi^q_{q-1}} J_{q-1}(V) \longrightarrow 0.$$

Here S^q is the bundle of symmetric tensors of order q. Recall that in a complex a composition of two consecutive maps is zero, and the exactness means that image of each map is the kernel of the following map.

Definition 2.6 Let $\mathcal{R}_q \subset J_q(V^0)$ be a sufficiently regular differential equation given by $\mathcal{R}_q = \ker \varphi_A$. The principal symbol σA of A is the map $S^q(T^*\mathcal{X}) \otimes V^0 \to V^1$ defined by $\sigma A = \varphi_A \varepsilon_q$.

To see that this actually coincides with the classical definition we need to introduce a coordinate system on $\mathcal{X}$. Then a linear qth order partial differential equation $\mathcal{R}_q$ is given by

$$Ay := \sum_{|\alpha| \le q} a_\alpha(x) D^\alpha y = f,$$

where $x \in U \subset \mathbb{R}^n$ and $a_\alpha(x)$ are of size $k \times m$. Fixing any one form ξ we get a bundle map $\sigma A(x,\xi) : V^0|_x \to V^1|_x$ which in coordinates is given by

$$\sigma A(x,\xi) = \sum_{|\alpha| = q} a_\alpha(x) \xi^\alpha .$$

A differential operator A is said to be *elliptic* if $\sigma A(x,\xi)$ is injective for all $x \in \mathcal{X}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$.

2.2 Compatibility Complexes

A linear partial differential operator $A : C^\infty(V^0) \to C^\infty(V^1)$ is *overdetermined* if there is a non-zero differential operator A^1 such that $A^1 A = 0$. Hence $A^1 f = 0$ is a necessary condition for the solvability of the system $Ay = f$.
A classical example of an overdetermined operator in $\mathcal{X} \subset \mathbb{R}^3$ is the gradient which maps a scalar function y to $\nabla y = (\partial y/\partial x_1, \partial y/\partial x_2, \partial y/\partial x_3)$. A necessary solvability condition for the system $\nabla\, y = f$ is

$$\nabla \times f = \big(\frac{\partial f^3}{\partial x_2} - \frac{\partial f^2}{\partial x_3}, \frac{\partial f^1}{\partial x_3} - \frac{\partial f^3}{\partial x_1}, \frac{\partial f^2}{\partial x_1} - \frac{\partial f^1}{\partial x_2}\big) = 0\ .$$

The operator $\nabla\times$ is itself overdetermined because $\nabla \cdot \nabla\times = 0$, where

$$\nabla \cdot h = \frac{\partial h^1}{\partial x_1} + \frac{\partial h^2}{\partial x_2} + \frac{\partial h^3}{\partial x_3}.$$

Hence setting $V^0 = \mathcal{X} \times \mathbb{R}$ and $V^1 = \mathcal{X} \times \mathbb{R}^3$ we get a complex

$$0 \longrightarrow C^\infty(V^0) \xrightarrow{\nabla} C^\infty(V^1) \xrightarrow{\nabla\times} C^\infty(V^1) \xrightarrow{\nabla\cdot} C^\infty(V^0) \longrightarrow 0\ \ .$$

The main problem in studying overdetermined systems consists in finding all solvability conditions for a given system $Ay = f$. The following definition explains the meaning of the words “all solvability conditions”.

Definition 2.7 Let $A^0 : C^\infty(V^0) \to C^\infty(V^1)$ be a differential operator. A differential operator $A^1 : C^\infty(V^1) \to C^\infty(V^2)$ is a *compatibility operator* for A^0 if

(i) $A^1 A^0 = 0$ and
(ii) for any differential operator $\tilde{A}^1 : C^\infty(V^1) \to C^\infty(\tilde{V}^2)$ such that $\tilde{A}^1 A^0 = 0$, there is a differential operator $T : C^\infty(V^2) \to C^\infty(\tilde{V}^2)$ such that $\tilde{A}^1 = TA^1$.

This idea leads naturally to

Definition 2.8 A complex

$$\mathcal{C} \quad : \quad 0 \longrightarrow C^\infty(V^0) \xrightarrow{A^0} C^\infty(V^1) \xrightarrow{A^1} C^\infty(V^2) \xrightarrow{A^2} \cdots$$

is a *compatibility complex* for A^0 if every differential operator A^i for $i \geq 1$ is a compatibility operator for A^{i-1}.

The following theorem gives the main result about the existence of compatibility complexes (see [3], [10] and [13] for more details).

Theorem 2.9 *Every sufficiently regular differential operator has a compatibility complex.*

2.3 Cochain Equivalence

We want to construct the compatibility complex for a given operator. However, it turns out that to do this we must first complete the system into involutive form, and then reduce it to a certain first order system. These other systems should be equivalent to the original one in order that this construction makes sense. The following definition gives the appropriate meaning of equivalence.

Definition 2.10 Two complexes

$$\mathcal{C} \quad : \quad \cdots \longrightarrow C^\infty(V^i) \xrightarrow{\Phi^i} C^\infty(V^{i+1}) \longrightarrow \cdots$$

$$\tilde{\mathcal{C}} \quad : \quad \cdots \longrightarrow C^\infty(\tilde{V}^i) \xrightarrow{\tilde{\Phi}^i} C^\infty(\tilde{V}^{i+1}) \longrightarrow \cdots$$

are *cochain equivalent* if the following conditions are satisfied:

1. there are differential operators M^i and N^i such that the following diagram commutes for all i

$$\begin{array}{ccc} C^\infty(V^i) & \xrightarrow{\Phi^i} & C^\infty(V^{i+1}) \\ M^i \downarrow \uparrow N^i & & M^{i+1} \downarrow \uparrow N^{i+1} \\ C^\infty(\tilde{V}^i) & \xrightarrow{\tilde{\Phi}^i} & C^\infty(\tilde{V}^{i+1}) \end{array}$$

2. there are differential operators Ψ^i and $\tilde{\Psi}^i$ such that for all i

$$\Psi^i\Phi^i + \Phi^{i-1}\Psi^{i-1} = \mathsf{id} - N^i M^i$$
$$\tilde{\Psi}^i\tilde{\Phi}^i + \tilde{\Phi}^{i-1}\tilde{\Psi}^{i-1} = \mathsf{id} - M^i N^i$$

Definition 2.11 Operators $\Phi : C^\infty(V^0) \to C^\infty(V^1)$ and $\tilde{\Phi} : C^\infty(\tilde{V}^0) \to C^\infty(\tilde{V}^1)$ are *cochain equivalent* if the complexes

$$0 \longrightarrow C^\infty(V^0) \xrightarrow{\Phi} C^\infty(V^1)$$

$$0 \longrightarrow C^\infty(\tilde{V}^0) \xrightarrow{\tilde{\Phi}} C^\infty(\tilde{V}^1)$$

are cochain equivalent.

If we know a compatibility complex for some operator, we can construct a compatibility complex for a cochain equivalent operator as follows.

Theorem 2.12 *Let Φ^0 and $\tilde{\Phi}^0$ be cochain equivalent differential operators. If there is a compatibility complex for $\tilde{\Phi}^0$, then there is also a compatibility complex for Φ^0. Moreover their compatibility complexes are cochain equivalent.*

Proof. Suppose that we know a compatibility complex for $\tilde{\Phi}^0$ and that a compatibility operator Φ^i, operators M^{i+1}, N^{i+1}, and Ψ^i have already been constructed as in Definition 2.10. Then we can construct Φ^{i+1} by the following formula.

$$\begin{aligned} \Phi^{i+1} &: C^\infty(V^{i+1}) \to C^\infty(V^{i+2}) = C^\infty(\tilde{V}^{i+2} \oplus V^{i+1}) \\ \Phi^{i+1} &= \big(\tilde{\Phi}^{i+1} M^{i+1}\big) \oplus \big(\mathsf{id} - N^{i+1}M^{i+1} - \Phi^i\Psi^i\big). \end{aligned} \tag{2.1}$$

For the details of the proof we refer to [13, p. 28]. □

3 Compatibility Complexes for Operators with Constant Coefficients in a Domain without Boundary

Consider a differential operator $A^0 : C^\infty(V^0) \to C^\infty(V^1)$ with constant coefficients on an open set $\mathcal{X} \subset \mathbb{R}^n$ where $V^0 = \mathcal{X} \times \mathbb{R}^{k_0}$ and $V^1 = \mathcal{X} \times \mathbb{R}^{k_1}$.

Let us introduce the *full symbol* of A^0:

$$\sigma_{\mathsf{F}}(A^0) = \sum_{|\alpha|\le q} a_\alpha \xi^\alpha .$$

Let $\mathbb{A} = \mathbb{K}[\xi_1, \dots, \xi_n]$ be a polynomial ring in n variables where $\mathbb{K}$ is some field of characteristic zero that contains the coefficients of the differential operator A^0. Denoting by $a^1, \dots, a^{k_1}$ the rows of $\sigma_{\mathsf{F}}(A_0)$ we may construct a free resolution of the module $M = \mathbb{A}^{k_0}/M_0$ where $M_0 = \langle a^1, \dots, a^{k_1}\rangle$:

$$0 \longrightarrow \mathbb{A}^{k_r} \xrightarrow{\mathsf{A}_r^T} \cdots \xrightarrow{\mathsf{A}_1^T} \mathbb{A}^{k_1} \xrightarrow{\sigma_{\mathsf{F}}(A^0)^T} \mathbb{A}^{k_0} \longrightarrow M \longrightarrow 0 \ .$$

Let A^i be the differential operator with the full symbol matrix $\sigma_{\mathsf{F}}(A^i) = \mathsf{A}_i$. In this case we say that the differential operator A^i is associated to the syzygy matrix A_i^T. Now a complex $\mathcal{C}$ consisting of trivial bundles $V^i = \mathcal{X} \times \mathbb{R}^{k_i}$ and operators A^i is said to be a *Hilbert complex*, if the operators A^i are associated to the syzygy matrices of the free resolution of $\mathbb{A}$-module M.

Theorem 3.1 *[13, p. 31] Let $\mathcal{C}$ be a complex of differential operators with constant coefficients. $\mathcal{C}$ is a compatibility complex for A^0 if and only if $\mathcal{C}$ is a Hilbert complex associated with the $\mathbb{A}$-module M.*

Hence the compatibility complex for a differential operator with constant coefficients on an open set in $\mathbb{R}^n$ can be constructively computed using Gröbner basis techniques.

Example 3.2 Consider the following system $Ay = (\nabla \times y, \nabla \cdot y)$. Let $b^1, \dots, b^4$ be the rows of the full symbol[1] of A

$$\sigma_{\mathsf{F}}(A) = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \\ \xi_1 & \xi_2 & \xi_3 \end{pmatrix}, \quad M_0 = \langle b^1, \dots, b^4\rangle.$$

Computing the syzygy of module M_0, we get

$$S = (\xi_1, \xi_2, \xi_3, 0), \quad M_1 = \langle S\rangle \ .$$

Computing the syzygy of M_1, we get $M_2 = 0$. Hence we have the following free resolution for $\mathbb{A}^3/M_0$:

$$0 \longrightarrow \mathbb{A}^1 \xrightarrow{S^T} \mathbb{A}^4 \xrightarrow{A^T} \mathbb{A}^3 \longrightarrow \mathbb{A}^3/M_0 \longrightarrow 0 \ .$$

Thus, the compatibility complex for A is

$$0 \longrightarrow C^\infty(\mathcal{X} \times \mathbb{R}^3) \xrightarrow{A} C^\infty(\mathcal{X} \times \mathbb{R}^4) \xrightarrow{(\nabla\cdot,0)} C^\infty(\mathcal{X} \times \mathbb{R}) \longrightarrow 0 \ .$$

[1] In fact in this case $\sigma_{\mathsf{F}}(A) = \sigma A$.

4 Compatibility Complexes for Boundary Problem Operators

From now on we suppose that $\mathcal{X}$ is a manifold with boundary $\mathcal{Y}$. To consider boundary value problems let us introduce two bundles W^0 and W^1 whose base manifold is the boundary $\mathcal{Y}$. The bundle $V^i|_{\mathcal{Y}} \to \mathcal{Y}$ is the restriction of $V^i \to \mathcal{X}$ to the boundary. If y is a section of $V^i \to \mathcal{X}$, then γy is the corresponding section of $V^i|_{\mathcal{Y}} \to \mathcal{Y}$. The map γ is called the trace map.

Definition 4.1 An operator $\mathcal{A} : C^\infty(V^0) \times C^\infty(W^0) \to C^\infty(V^1) \times C^\infty(W^1)$ of the form

$$\mathcal{A}(y,w) = \begin{pmatrix} \mathcal{A}_{1,1} & 0 \\ \gamma\mathcal{A}_{2,1} & \mathcal{A}_{2,2} \end{pmatrix} \begin{pmatrix} y \\ w \end{pmatrix},$$

where $\mathcal{A}_{i,j}$ are differential operators, is called a *boundary problem operator*.

If $W^0 = 0$, we obtain in this way an operator $\mathcal{A}(y) = (Ay, \gamma By)$ which defines a classical boundary problem on $\mathcal{X}$.

It turns out that one can construct the compatibility complex for a boundary problem operator $\mathcal{A} = (A, \gamma B)$ using a certain equivalent first order system.

Definition 4.2 A differential operator $A : C^\infty(V^0) \to C^\infty(V^1)$ is *normalised* if

(i) A is a first order operator;
(ii) A is involutive;
(iii) the principal symbol $\sigma A : T^*\mathcal{X} \otimes V^0 \to V^1$ is surjective.

Condition (iii) means that there are no (explicit or implicit) algebraic (i.e., non-differential) relations between dependent variables in the system. If such relations exist, then we may use them to reduce the number of dependent variables.

Theorem 4.3 *Every sufficiently regular operator A can be transformed in a finite number of steps into an equivalent normalised operator.*

Definition 4.4 A boundary problem operator $\mathcal{A}$ is normalised if $\mathcal{A}_{1,1}$ is normalised and $\gamma\mathcal{A}_{2,1}$ contains only differentiation in directions tangent to the boundary.

Theorem 4.5 *Every boundary problem operator $\mathcal{A}$ whose component $\mathcal{A}_{1,1}$ is sufficiently regular is cochain equivalent to a normalised boundary problem operator.*

For the proofs of the above theorems we refer to [13].

To construct compatibility operators, we introduce the tangent part of a first order differential operator $A : C^\infty(V^0) \to C^\infty(V^1)$. Let us choose a coordinate system $x = (x_1, \dots, x_n)$ such that the boundary is given by the equation $x_n = 0$. Then in these coordinates there is a part of A which contains differentiations only with respect to $x_1, \dots, x_{n-1}$. This part is denoted by $A^\tau : C^\infty(V^0|_\mathcal{Y}) \to C^\infty(V^{1^\tau})$ where V^{1^τ} is a certain bundle over $\mathcal{Y}$, and it is called the *tangent part* of A. It can be shown that if A is normalised, then so is A^τ [3].

If we have a system $Ay = f$, then for the tangent part we get a system $A^\tau y = f^\tau$. This defines a projection

$$\mathsf{pr}^\tau : V^1 \to V^{1^\tau} \quad , \quad \mathsf{pr}^\tau(f) = f^\tau .$$

Now we can rewrite any normalised boundary problem operator in the form

$$\begin{aligned} \mathcal{A} : C^\infty(V^0) \times C^\infty(W^0) &\to C^\infty(V^1) \times C^\infty(W^1) \\ \mathcal{A}(y, w) &= \begin{pmatrix} \mathcal{A}_{1,1} & 0 \\ \mathcal{A}_{2,1}\gamma & \mathcal{A}_{2,2} \end{pmatrix} \begin{pmatrix} y \\ w \end{pmatrix} \end{aligned} \tag{4.1}$$

where $\mathcal{A}_{2,1}$ is a differential operator on the boundary $\mathcal{Y}$. Then on $\mathcal{Y}$ we define a differential operator $\mathcal{A}^\tau : C^\infty(V^0|_\mathcal{Y}) \times C^\infty(W^0) \to C^\infty(V^{1^\tau}) \times C^\infty(W^1)$,

$$\mathcal{A}^\tau(y^\tau, w) = \begin{pmatrix} \mathcal{A}^\tau_{1,1} & 0 \\ \mathcal{A}_{2,1} & \mathcal{A}_{2,2} \end{pmatrix} \begin{pmatrix} y^\tau \\ w \end{pmatrix} \tag{4.2}$$

Definition 4.6 A normalised boundary problem operator $\mathcal{A}$ is *regular* if the differential operator $\mathcal{A}^\tau$ is sufficiently regular.

Definition 4.7 A boundary problem operator $\mathcal{A}$ is *regular* if the differential operators $\mathcal{A}_{1,1}$ and $\mathcal{A}^\tau_{\mathsf{norm}}$ are sufficiently regular where $\mathcal{A}^\tau_{\mathsf{norm}}$ corresponds to an equivalent normalised operator.

We start the construction of a compatibility operator. It suffices to consider regular normalised boundary problem operators. Then, we can use Theorems 4.5 and 2.12, which enable us to construct a compatibility operator for an arbitrary regular boundary problem operator.

Let $\mathcal{A}$ be a regular normalised boundary problem operator given by (4.1), $\mathcal{A}^\tau_{1,1}$ be the tangent part of $\mathcal{A}_{1,1}$ and $\mathcal{A}^\tau$ the operator defined by (4.2). As $\mathcal{A}^\tau$ is sufficiently regular, by Theorem 2.9 it has a compatibility operator $\mathcal{A}^{\tau 1}$ which can always be written in the form

$$\mathcal{A}^{\tau 1}(f^\tau, g) = \begin{pmatrix} \mathcal{A}^{\tau 1}_{1,1} f^\tau \\ \Upsilon^\tau(f^\tau, g) \end{pmatrix} . \tag{4.3}$$

Here $\mathcal{A}_{1,1}^{\tau 1}$ is a compatibility operator for $\mathcal{A}_{1,1}^{\tau}$ and Υ^τ does not contain any relations involving only the components of f^τ. Let us then finally define

$$\mathcal{A}^1 \ : \ C^\infty(V^1) \times C^\infty(W^1) \to C^\infty(V^2) \times C^\infty(W^2),$$
$$\mathcal{A}^1(f,g) = \begin{pmatrix} \mathcal{A}_{1,1}^1 f \\ \Upsilon^\tau(\mathsf{pr}^\tau f, g) \end{pmatrix}, \tag{4.4}$$

where $\mathcal{A}_{1,1}^1$ is a compatibility operator for $\mathcal{A}_{1,1}$. We will need the following important result [3, p. 40].

Theorem 4.8 *Let $\mathcal{A}$ be a regular normalised boundary operator whose component $\mathcal{A}_{1,1}$ is elliptic. Then $\mathcal{A}^1$ defined by* (4.4) *is a compatibility operator for $\mathcal{A}$.*

The operator $\mathcal{A}^1$ is itself regular and normalised. This together with ellipticity of $\mathcal{A}_{1,1}$ enables us to construct the whole compatibility complex for $\mathcal{A}$

$$0 \longrightarrow C^\infty(V^0) \times C^\infty(W^0) \xrightarrow{\mathcal{A}} C^\infty(V^1) \times C^\infty(W^1) \xrightarrow{\mathcal{A}^1} \cdots$$

If $\mathcal{A}$ is regular but not normalised, then it needs to be replaced by an equivalent normalised operator for which the compatibility complex is constructed. But then by Theorem 2.12 we can construct the compatibility complex for $\mathcal{A}$ using the compatibility complex of the corresponding normalised operator.

5 Computations

Here we show that on each step of the construction of a compatibility operator for a normalised boundary problem operator one may effectively use Gröbner basis computations.

- **Computation of the tangent part A^τ of a differential operator A.**

  Let $M \subset \mathbb{A}^m$ be the module generated by the rows of the full symbol of A. We choose a product ordering such that ξ_n is bigger than all other ξ_i. Then we define a TOP module ordering using this ordering and compute the Gröbner basis of M. Now A^τ is defined by the elements of the Gröbner basis that do not contain ξ_n. This follows from the fact [1, p. 156] that if G is a Gröbner basis for M, then $G \cap \tilde{\mathbb{A}}^m$ is a Gröbner basis for $M \cap \tilde{\mathbb{A}}^m$ where $\tilde{\mathbb{A}} = \mathbb{K}[\xi_1, \dots, \xi_{n-1}]$.
- **Computation of a compatibility operator $\mathcal{A}^{\tau 1}$ for $\mathcal{A}^\tau$ defined in** (4.2).

  Since $\mathcal{A}^\tau$ is a differential operator on the manifold $\mathcal{Y}$ without boundary, one can compute its compatibility operator by computing the syzygy module of the module generated by the rows of the full symbol of $\mathcal{A}^\tau$, see Theorem 3.1.

- **Computation of operator Υ^τ in** (4.3)

  In the previous step we computed $\mathcal{A}^{\tau 1}$; now we would like to eliminate rows which contain only f^τ. To do this we choose a POT module ordering (and any convenient monomial ordering) and compute the Gröbner basis G of the module generated by the rows of $\sigma_{\mathsf{F}}\big(\mathcal{A}^{\tau 1}\big)$. The elements of G correspond to differential operators which operate on (f^τ, g). The full symbol of Υ^τ is now obtained by discarding those elements whose corresponding differential operators operate on f^τ only.

  The correctness of this construction follows from the following fact. Let M be a submodule of $\mathbb{A}^m = \mathbb{A}^i \oplus \mathbb{A}^{m-i}$. We choose a POT module ordering for $\mathbb{A}^m$, and any monomial ordering in $\mathbb{A}$. Then if G is a Gröbner basis for M, then $G \cap \mathbb{A}^{m-i}$ is a Gröbner basis for $M \cap \mathbb{A}^{m-i}$ [5, p. 177].

6 Example

Let us construct a compatibility complex for the stationary Stokes problem in two dimensions. Consider the boundary problem in $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x_2 > 0\}$

$$A \;:\; \begin{cases} -\Delta u + \nabla p = 0, \\ \nabla \cdot u = 0 \end{cases} \quad \text{in } \mathcal{X} = \mathbb{R}^2_+, \quad B \;:\; \begin{cases} -u^2_{20} + p_{01} = 0, \\ u^1_{11} = 0 \end{cases} \quad \text{on } \mathcal{Y} = \partial\mathbb{R}^2_+,$$

where $u = (u^1, u^2)$ is the velocity field and p is the pressure. Here we have used for the derivatives the notation

$$u^i_{k\ell} = \frac{\partial^{k+\ell} u^i}{\partial x_1^k \partial x_2^\ell}$$

By completing the above system to the involutive form we arrive at an overdetermined system

$$A_0 : \quad \begin{cases} -\Delta u + \nabla p &= 0, \\ \Delta p &= 0, \\ \nabla \cdot u &= 0, \\ u^1_{20} + u^2_{11} &= 0, \\ u^1_{11} + u^2_{02} &= 0 \end{cases} \tag{6.1}$$

Introducing nine new variables (unknown functions)

$$\begin{aligned} v^{1,00} &= u^1, & v^{1,10} &= u^1_{10}, & v^{1,01} &= u^1_{01}, \\ v^{2,00} &= u^2, & v^{2,10} &= u^2_{10}, & v^{2,01} &= u^2_{01}, \\ v^{3,00} &= p, & v^{3,10} &= p_{10}, & v^{3,01} &= p_{01} \end{aligned}$$

and substituting them into (6.1), and also adding the compatibility equations we get the first order system

$$A_0' : \begin{cases} -v_{10}^{1,10} - v_{01}^{1,01} + v^{3,10} &= 0, \\ -v_{10}^{2,10} - v_{01}^{2,01} + v^{3,01} &= 0, \\ v_{10}^{3,10} + v_{01}^{3,01} &= 0, \\ v^{1,10} + v^{2,01} &= 0, \\ v_{10}^{1,10} + v_{10}^{2,01} &= 0, \\ v_{01}^{1,10} + v_{01}^{2,01} &= 0, \\ v_{10}^{j,00} - v^{j,10} &= 0, \\ v_{01}^{j,00} - v^{j,01} &= 0, \\ v_{01}^{j,10} - v_{10}^{j,01} &= 0 \end{cases}$$

for $j = 1, 2, 3$. This system is not normalised since there is an algebraic relation $v^{1,10} + v^{2,01} = 0$ between the dependent variables. Using this relation we can now eliminate the unknown function $v^{2,01}$ from the system and obtain the following normalised system

$$A_0'' : \begin{cases} -v_{10}^{1,10} - v_{01}^{1,01} + v^{3,10} &= 0, \\ v_{01}^{1,10} - v_{10}^{2,10} + v^{3,01} &= 0, \\ v_{10}^{3,10} + v_{01}^{3,01} &= 0, \\ v_{10}^{j,00} - v^{j,10} &= 0, \\ v_{10}^{2,00} - v^{2,10} &= 0, \\ v_{01}^{j,00} - v^{j,01} &= 0, \\ v_{01}^{2,00} + v^{1,10} &= 0, \\ v_{01}^{j,10} - v_{10}^{j,01} &= 0, \\ v_{10}^{1,10} + v_{01}^{2,10} &= 0 \end{cases}$$

for $j = 1, 3$. Finally, substituting the new unknown functions in the boundary conditions, we obtain

$$B'' : \begin{cases} -v_{10}^{2,10} + v^{3,01} &= 0, \\ v_{10}^{1,01} &= 0. \end{cases}$$

Hence it follows that the classical boundary problem operator $\mathcal{A}'' = (A_0'', B'')$ is normalised. Using Gröbner basis computations, we find the tangent part of A_0''

$$A_0''^{\tau} : \begin{cases} v_{10}^{2,10} - v_{10}^{1,01} - v^{3,01} &= 0, \\ v_{10}^{3,00} - v^{3,10} &= 0, \\ v_{10}^{2,00} - v^{2,10} &= 0, \\ v_{10}^{1,00} - v^{1,10} &= 0. \end{cases}$$

Let us compute a compatibility operator for the operator $\mathcal{A}''^{\tau} = (A_0''^{\tau}, B'')$ defined on the boundary $\mathcal{Y}$. Computing the syzygy module for the module generated by the rows of the full symbol matrix of the operator $\mathcal{A}''^{\tau}$, we get

$$\mathcal{A}''^{\tau 1}(f^{\tau}, g) = f^{1^{\tau}} + g^1 + g^2$$

where $f^{\tau} = (f^{1^{\tau}}, \ldots, f^{4^{\tau}})$ and $g = (g^1, g^2)$. Now let us compute a compatibility operator for A_0''. Computing the syzygy module of the module generated by the rows of the full symbol of A_0'', we have

$$A_0''^1(f'') = \begin{pmatrix} f_{01}''^4 - f_{10}''^7 + f''^{10} \\ f_{01}''^5 - f_{10}''^8 + f''^{11} \\ f_{01}''^6 - f_{10}''^9 + f''^{12} \\ -f_{10}''^1 - f_{01}''^2 + f''^3 + f_{01}''^{10} - f_{10}''^{12} \end{pmatrix}$$

where $f'' = (f''^1, \ldots, f''^{12})$. Note that the projection $\mathsf{pr}^{\tau} : V''^1 \to V''^{1^{\tau}}$ is given by

$$\mathsf{pr}^{\tau}(f'') = (f''^{10}|_{\mathcal{Y}} - f''^2|_{\mathcal{Y}}, f''^5|_{\mathcal{Y}}, f''^6|_{\mathcal{Y}}, f''^4|_{\mathcal{Y}}).$$

Using (4.4) we find a compatibility operator for the normalised boundary problem operator $\mathcal{A}''$,

$$\mathcal{A}''^1(f'', g) = \begin{pmatrix} f_{01}''^4 - f_{10}''^7 + f''^{10} \\ f_{01}''^5 - f_{10}''^8 + f''^{11} \\ f_{01}''^6 - f_{10}''^9 + f''^{12} \\ -f_{10}''^1 - f_{01}''^2 + f''^3 + f_{01}''^{10} - f_{10}''^{12} \\ f''^{10}|_{\mathcal{Y}} - f''^2|_{\mathcal{Y}} + g^1 + g^2 \end{pmatrix}.$$

Now we compute that the tangent part of $\mathcal{A}''^1_{1,1}$ is the zero operator. Then (4.2) implies that

$$\mathcal{A}''^{1^{\tau}}(f''^{\tau}, g) = \begin{pmatrix} 0 \\ f''^{10^{\tau}} - f''^{2^{\tau}} + g^1 + g^2 \end{pmatrix}.$$

It is clear that the compatibility operator for $\mathcal{A}''^{1^{\tau}}$ is the zero operator. Since the compatibility operator for $\mathcal{A}''^1_{1,1}$ is equal to zero, the compatibility operator for $\mathcal{A}''^1$ is the zero operator as well. Hence, we arrive at the following compatibility complex for the normalised boundary problem operator $\mathcal{A}''$,

$$\begin{array}{ccccc} 0 \longrightarrow C^{\infty}(V''^0) & \xrightarrow{\mathcal{A}''} & C^{\infty}(V''^1) \times C^{\infty}(W^1) & & \\ & & \Big\downarrow \mathcal{A}''^1 & & \\ & & C^{\infty}(V''^2) \times C^{\infty}(W''^2) & \longrightarrow & 0 \end{array}$$

Now we will construct a compatibility complex for the original boundary problem operator $\mathcal{A} = (A, B)$. First simple computations show that boundary problem operators

$\mathcal{A}''$ and $\mathcal{A}$ are cochain equivalent with the following operators in Definition 2.11,

$$\begin{array}{ccccc} 0 & \longrightarrow & C^\infty(V^0) & \xrightarrow{\mathcal{A}} & C^\infty(V^1)\times C^\infty(W^1) \\ & & M^0 \downarrow \uparrow N^0 & & M^1 \downarrow \uparrow N^1 \\ 0 & \longrightarrow & C^\infty(V''^0) & \xrightarrow{\mathcal{A}''} & C^\infty(V''^1)\times C^\infty(W^1) \end{array}$$

$$\begin{aligned}
&M^0(u^1,u^2,p) = (u^1,u^2,p,u^1_{10},u^2_{10},p_{10},u^1_{01},p_{01}),\\
&N^0(v^{1,00},v^{2,00},v^{3,00},v^{1,10},v^{2,10},v^{3,10},v^{1,01},v^{3,01}) = (v^{1,00},v^{2,00},v^{3,00}),\\
&M^1(f^1,f^2,f^3,g^1,g^2) =\\
&\qquad (f^1,f^2+f^3_{01},f^1_{10}+f^2_{01}+\Delta f^3,0,0,0,0,0,f^3,0,0,f^3_{10},g^1,g^2),\\
&N^1(f''^1,\dots,f''^{12},g^1,g^2) =\\
&\qquad (f''^1-f''^4_{10}-f''^7_{01}+f''^5, f''^2-f''^6_{10}-f''^9_{01}+f''^8, f''^4+f''^9,g^1,g^2),\\
&\Psi^0(f^1,f^2,f^3,g^1,g^2) = 0,\\
&\tilde{\Psi}^0(f''^1,\dots,f''^{12},g^1,g^2) = (0,0,0,-f''^4,-f''^6,-f''^5,-f''^7,-f''^8).
\end{aligned}$$

Then using formula (2.1), we define compatibility operator for $\mathcal{A}$ by

$$\mathcal{A}^1 \;:\; C^\infty(V^1)\times C^\infty(W^1) \to C^\infty(V''^2\oplus V^1)\times C^\infty(W''^2\oplus W^1),$$
$$\mathcal{A}^1 = \mathcal{A}''^1 M^1 \oplus (\mathsf{id} - N^1M^1).$$

Computing $\mathcal{A}^1$, we get

$$\mathcal{A}^1(f^1,f^2,f^3,g^1,g^2) = \begin{pmatrix} 0 & 0 & 0 & 0 & -f^2|_{\mathcal{Y}} - f^3_{01}|_{\mathcal{Y}} + g^1 + g^2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, we arrive at the following compatibility complex for the original boundary problem operator $\mathcal{A}$,

$$\begin{array}{ccccc} 0 & \longrightarrow & C^\infty(V^0) & \xrightarrow{\mathcal{A}} & C^\infty(V^1)\times C^\infty(W^1) \\ & & & & \downarrow \mathcal{A}^1 \\ & & & & C^\infty(V''^2\oplus V^1)\times C^\infty(W''^2\oplus W^1) \longrightarrow 0\,. \end{array}$$

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Author information

Katsiaryna Krupchyk, Department of Physics and Mathematics, University of Joensuu, P.O. Box 111, FI-80101 Joensuu, Finland.
Email: katya.krupchyk@joensuu.fi

Jukka Tuomela, Department of Physics and Mathematics, University of Joensuu, P.O. Box 111, FI-80101 Joensuu, Finland.
Email: jukka.tuomela@joensuu.fi