Differential Equations, Vol. 41, No. 2, 2005, pp. 184–194. Translated from Differentsial'nye Uravneniya, Vol. 41, No. 2, 2005, pp. 177–185.

ORDINARY DIFFERENTIAL EQUATIONS

The Construction of a Pfaff System with Arbitrary Piecewise Continuous Characteristic Power-Law Functions

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Received March 24, 2004

Consider the linear Pfaff system

$$\partial x/\partial t_i = A_i(t)x, \qquad x \in R^n, \qquad t = (t_1, t_2) \in R^2_{>1}, \qquad i = 1, 2, \qquad n \in N, \qquad (1_n)$$

with bounded continuously differentiable matrix functions $A_i(t)$ satisfying the complete integrability condition [1, pp. 43–44; 2, pp. 21–24]

$$\partial A_1(t)/\partial t_2 + A_1(t)A_2(t) = \partial A_2(t)/\partial t_1 + A_2(t)A_1(t), \qquad t \in R^2_{>1}.$$

Let $p = p[x] \in R^2$ and $\lambda = \lambda[x] \in R^2$ be the lower characteristic vector [3] and the characteristic vector [4], respectively, of a nontrivial solution $x : R^2_{>1} \to R^n \backslash \{0\}$ of system (1_n), and let $P_x = \{p[x]\}$ and $\Lambda_x = \{\lambda[x]\}$ be the lower characteristic set and the characteristic set of the same solution; they are bounded and closed and can be represented [3, 4], respectively, by monotone decreasing concave and convex curves $p_2 = \varphi(p_1) : [\alpha_1, \alpha_2] \to [\beta_1, \beta_2]$ and $\lambda_2 = f(\lambda_1) : [a_1, a_2] \to [b_1, b_2]$ on the plane R^2.

We introduce analogs of the Demidovich characteristic degree [5] of a function of a single variable, namely, the lower characteristic degree [6] $\underline{d} = \underline{d}_x(p) \in R^2$ and the upper characteristic degree [7] $\bar{d} = \bar{d}_x(\lambda) \in R^2$ of the solution $x \neq 0$, given by the conditions

$$\underline{\ln}_x(p, \underline{d}) \equiv \varliminf_{t\to\infty} \frac{\ln\|x(t)\| - (p, t) - (\underline{d}, \ln t)}{\|\ln t\|} = 0,$$
$$\underline{\ln}_x(p, \underline{d} + \varepsilon e_i) < 0, \qquad \forall \varepsilon > 0, \qquad i = 1, 2,$$
$$\overline{\ln}_x(\lambda, \bar{d}) \equiv \varlimsup_{t\to\infty} \frac{\ln\|x(t)\| - (\lambda, t) - (\bar{d}, \ln t)}{\|\ln t\|} = 0,$$
$$\overline{\ln}_x(\lambda, \bar{d} - \varepsilon e_i) > 0, \qquad \forall \varepsilon > 0, \qquad i = 1, 2,$$

where $e_i = (2-i, i-1)$ and $\ln t \equiv (\ln t_1, \ln t_2) \in R^2_+$, as well as the lower degree set $\underline{D}_x(p) = \{\underline{d}_x(p)\}$ and the upper degree set $\bar{D}_x(\lambda) = \{\bar{d}_x(\lambda)\}$.

By [6], if for an interior point $p = (p_1, \varphi(p_1)) \in P_x$, $p_1 \in (\alpha_1, \alpha_2)$, and for an exterior point $\lambda = (\lambda_1, f(\lambda_1)) \in \Lambda_x$, $\lambda_1 \in (a_1, a_2)$, there exist finite limits $\underline{c}_x(p_1) \equiv \sqrt{2}\underline{ln}_x((p_1, \varphi(p_1)), 0)$ and $\bar{c}_x(\lambda_1) \equiv \sqrt{2}\,\overline{\ln}_x((\lambda_1, f(\lambda_1)), 0)$, respectively, then the individual interior lower degree set $\underline{D}_x(p)$ and upper degree set $\bar{D}_x(\lambda)$ of the solution $x \neq 0$ are the lines $\underline{d}_1 + \underline{d}_2 = \underline{c}_x(p_1)$ and $\bar{d}_1 + \bar{d}_2 = \bar{c}_x(\lambda_1)$, respectively, on the plane R^2.

It was shown in [8] that the left boundary lower degree set $\underline{D}_x(p')$, $p' = (\alpha_1, \beta_2)$, the right boundary lower degree set $\underline{D}_x(p'')$, $p'' = (\alpha_2, \beta_1)$, the left boundary upper degree set $\bar{D}_x(\lambda')$, $\lambda' = (a_1, b_2)$, and the right boundary upper degree set $\bar{D}_x(\lambda'')$, $\lambda'' = (a_2, b_1)$, of the solution $x \neq 0$ of system (1_n) do not necessarily coincide with any line on the two-dimensional plane and even need not contain a rectilinear segment.

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Therefore, it is natural to introduce the lower characteristic degree function $\underline{c}_x(p_1)$ and the upper characteristic degree function $\bar{c}_x(\lambda_1)$ of the solution $x \neq 0$ of system (1_n) defined on the intervals (α_1, α_2) and (a_1, a_2), respectively.

The investigation of these functions was initiated in [6]; more precisely, completely integrable Pfaff systems (1_n) with infinitely differentiable bounded coefficients such that the characteristic degree functions $\underline{c}_x(p_1)$ and $\bar{c}_x(\lambda_1)$ of nontrivial solutions of these systems coincide with given functions were constructed for some piecewise constant functions and some continuous functions. The next step in the description of the functions $\underline{c}_x(p_1)$ and $\bar{c}_x(\lambda_1)$ is to implement an arbitrary given piecewise continuous function by characteristic degree functions of some nontrivial solution of some system (1_n).

By [9, p. 106 of the Russian translation], a piecewise continuous function on an interval is defined as a function continuous everywhere on that interval except for finitely many jump discontinuities.

Theorem 1. *For an arbitrary positive integer n and for an arbitrary piecewise continuous function $g : [\alpha_1, \alpha_2] \to R$, there exists a completely integrable Pfaff system (1_n) with infinitely differentiable bounded coefficients such that, for each nontrivial solution $x : R^2_{>1} \to R^n \backslash \{0\}$ of this system, the domain of the curve P_x coincides with the interval $[\alpha_1, \alpha_2]$ and*

$$\underline{c}_x(p_1) = g(p_1), \qquad \forall p_1 \in (\alpha_1, \alpha_2). \tag{2}$$

Proof. Note that it suffices to prove the desired assertion for $n = 1$. Indeed, having constructed a one-dimensional completely integrable Pfaff equation

$$\partial x/\partial t_i = a_i(t)x, \qquad x \in R, \qquad t \in R^2_{>1}, \qquad i = 1, 2, \tag{1_1}$$

with infinitely differentiable bounded coefficients such that the lower characteristic degree function $\underline{c}_{x_1}(p_1[x_1])$ of a nontrivial solution $x_1 : R^2_{>1} \to R\backslash\{0\}$ coincides with the function $g(p_1[x_1])$ for all $p_1[x_1] \in (\alpha_1, \alpha_2)$, one can choose the desired n-dimensional system (1_n) in the form of the diagonal system with coefficient matrix $A_i(t) = \operatorname{diag}[a_i(t), \ldots, a_i(t)]$, $i = 1, 2$, of order n. Then $X(t) = x_1(t)E_n$, where E_n is the identity matrix of order n, is the principal solution matrix of system (1_n), and each nontrivial solution $x : R^2_{>1} \to R^n\backslash\{0\}$ of this system can be represented in the form $x(t) = X(t)c = x_1(t)c$ with some $c \in R^n\backslash\{0\}$. Therefore, the lower characteristic vectors $p[x] = p[x_1]$ and hence the lower characteristic degree functions $\underline{c}_x(p_1[x]) = \underline{c}_{x_1}(p_1[x_1])$ of this solution and the solution x_1 of Eq. (1_1) coincide.

First, consider the case of an interval $[\alpha_1, \alpha_2]$ on the line R such that $\alpha_1 < \alpha_2 < 0$.

1. CONSTRUCTION OF THE SOLUTION

We construct a solution x of the Pfaff equation (1_1) in the form

$$\ln x(t) = \ln \phi(t) + \nu(t) \ln \psi(t), \qquad t \in R^2_{>1}. \tag{3}$$

The function ϕ is defined so as to ensure that the domain of the curve P_ϕ of its lower characteristic set coincides with the interval $[\alpha_1, \alpha_2]$ and all points $p[\phi]$ of the lower characteristic set P_ϕ are realized in different directions $\theta_t \equiv t_2/t_1$ depending on $p[\phi]$. We define the function ν on the basis of the function $g\left(-\sqrt{\theta_t}\right)$ so as to ensure that the solution x satisfies relation (2), $\ln x$ is an infinitely differentiable function, and its derivatives $\partial \ln x(t)/\partial t_i$, $i = 1, 2$, are bounded.

We use the infinitely differentiable standard function [10, p. 54 of the Russian translation]

$$e_{01}(\tau; \tau_1, \tau_2) = \begin{cases} \exp\left\{-(\tau - \tau_1)^{-2} \exp\left[-(\tau_2 - \tau)^{-2}\right]\right\} & \text{for} \quad \tau \in (\tau_1, \tau_2) \\ [1 + \operatorname{sgn}(\tau - 2^{-1}(\tau_1 + \tau_2))]/2 & \text{for} \quad \tau \notin (\tau_1, \tau_2), \end{cases}$$

$-\infty < \tau_1 < \tau_2 < +\infty$, and define infinitely differentiable functions ϕ and ψ by the relations

$$\begin{aligned} \ln \phi(t) = {} & -2\sqrt{t_1 t_2} + \left(2\sqrt{t_1 t_2} + t_2/\alpha_1\right) e_{01}\left(\theta_t/\left(2\alpha_1^2\right); 1/2, 1\right) \\ & + \left(2\sqrt{t_1 t_2} + \alpha_2 t_1\right)\left[1 - e_{01}\left(\theta_t/\alpha_2^2; 1/2, 1\right)\right], \end{aligned} \tag{4}$$

$$\ln \psi(t) = e_{01}\left(\theta_t/\alpha_2^2; 1/2, 1\right)\left[1 - e_{01}\left(\theta_t/\left(2\alpha_1^2\right); 1/2, 1\right)\right] \ln t_1, \qquad t \in R^2_{>1}. \tag{5}$$

Suppose that, on the interval (α_1, α_2), the function g has k points

$$\alpha_1 < \Delta_1 < \Delta_2 < \cdots < \Delta_k < \alpha_2$$

of discontinuity. For convenience, we set $\Delta_0 = \alpha_1$, $\Delta_{k+1} = \alpha_2$, and $\tau_j = \Delta_j - \Delta_{j-1}$, $j = 1, \ldots, k+1$. On the basis of the function g, we introduce the new functions

$$g_j : [\Delta_{j-1}, \Delta_j] \to R, \qquad j = 1, \ldots, k+1,$$

given by the relations

$$g_j(\Delta) = g(\Delta), \qquad \Delta \in (\Delta_{j-1}, \Delta_j), \qquad g_j(\Delta_{j-1}) = g(\Delta_{j-1} + 0), \qquad g_j(\Delta_j) = g(\Delta_j - 0).$$

Obviously, each function g_j is continuous on the closed interval $[\Delta_{j-1}, \Delta_j]$, $j = 1, \ldots, k+1$. By the Weierstrass approximation theorem, for each function $g_j(\Delta)$, there exists a sequence $\{P_l^{(j)}(\Delta)\}_{l \in N}$ of algebraic polynomials uniformly converging on $[\Delta_{j-1}, \Delta_j]$ to the function $g_j(\Delta)$. By virtue of the Weierstrass theorem on continuous functions on an interval, the following notation is well defined:

$$\mu_l = \max_{j=1,\ldots,k+1} \left(\max_{\Delta \in [\Delta_{j-1}, \Delta_j]} \left| P_l^{(j)}(\Delta) \right| \right) < +\infty,$$

$$\varrho_l = \max_{j=1,\ldots,k+1} \left(\max_{\Delta \in [\Delta_{j-1}, \Delta_j]} \left| dP_l^{(j)}(\Delta)/d\Delta \right| \right) < +\infty.$$

We use some value $\delta_0 > (\max_{j=0,\ldots,k+1} |g(\Delta_j)|)^4 (1 + 2\alpha_1^2) + 1$ and introduce the numbers

$$\gamma_l = (\delta_{l-1} + (1 + \varrho_l^2 + \mu_l^4)(1 + \alpha_1^2)) l^2, \qquad \delta_l = \gamma_l e^2, \qquad l \in N.$$

We split the quadrant $R_{>1}^2$ (that is, the domain of the solution x to be constructed) into disjoint "basic" strips

$$\Pi(l) = \left\{ t \in R_{>1}^2 : \ \delta_l \le t_1 + t_2 \equiv \zeta(t) \le \gamma_{l+1} \right\},$$

"transition" strips

$$\tilde{\Pi}(l) = \left\{ t \in R_{>1}^2 : \gamma_l < \zeta(t) < \delta_l \right\}, \qquad l \in N,$$

and the triangle $T = \{t \in R_{>1}^2 : \zeta(t) \le \gamma_1\}$.

Let us now proceed to the construction of the function ν. We first define auxiliary functions ν_l, $l \in N$, as follows:

$$\nu_l(t) = P_l^{(j)}\left(-\sqrt{\theta_t}\right), \qquad \theta_t \in \left[\left(\Delta_j - \tau_j/\left(2\sqrt{l}\right)\right)^2, \left(\Delta_{j-1} + \tau_j/\left(2\sqrt{l}\right)\right)^2 \right], \quad j = 1, \ldots, k+1, \tag{6_1}$$

$$\nu_l(t) = g(\Delta_j), \qquad \theta_t \in \left[\left(\Delta_j + \tau_{j+1}/\left(4\sqrt{l}\right)\right)^2, \left(\Delta_j - \tau_j/\left(4\sqrt{l}\right)\right)^2 \right], \quad j = 1, \ldots, k, \tag{6_2}$$

$$\begin{aligned} \nu_l(t) = {} & P_l^{(j)}\left(-\sqrt{\theta_t}\right) + \left[g(\Delta_j) - P_l^{(j)}\left(-\sqrt{\theta_t}\right) \right] \\ & \times \left[1 - e_{01}\left(2\sqrt{l}\left(\Delta_j + \sqrt{\theta_t}\right) \Big/ \tau_j; 1/2, 1 \right) \right], \\ & \theta_t \in \left(\left(\Delta_j - \tau_j \Big/ \left(4\sqrt{l}\right)\right)^2, \left(\Delta_j - \tau_j \Big/ \left(2\sqrt{l}\right)\right)^2 \right), \qquad j = 1, \ldots, k+1, \end{aligned} \tag{6_3}$$

$$\begin{aligned} \nu_l(t) = {} & g(\Delta_j) + \left[P_l^{(j+1)}\left(-\sqrt{\theta_t}\right) - g(\Delta_j) \right] \\ & \times \left[1 - e_{01}\left(2\sqrt{l}\left(\Delta_j + \sqrt{\theta_t}\right) \Big/ \tau_{j+1}; -1, -1/2 \right) \right], \\ & \theta_t \in \left(\left(\Delta_j + \tau_{j+1} \Big/ \left(2\sqrt{l}\right)\right)^2, \left(\Delta_j + \tau_{j+1} \Big/ \left(4\sqrt{l}\right)\right)^2 \right), \qquad j = 0, \ldots, k, \end{aligned} \tag{6_4}$$

$$\nu_l(t) = g(\alpha_1), \qquad \theta_t \geq \left(\alpha_1 + \tau_1 \Big/ \left(4\sqrt{l}\right)\right)^2, \tag{6_5}$$

$$\nu_l(t) = g(\alpha_2), \qquad \theta_t \leq \left(\alpha_2 - \tau_{k+1} \Big/ \left(4\sqrt{l}\right)\right)^2. \tag{6_6}$$

Relations (6_1)–(6_6) define the function ν_l in the entire quadrant $R^2_{>1}$. We set the desired function ν in the "basic" strips $\Pi(l)$ to be equal to the function ν_l; i.e.,

$$\nu(t) = \nu_l(t), \qquad t \in \Pi(l), \qquad l \in N. \tag{7_1}$$

In the "transition" strips, we pass from the function ν_{l-1} to the function ν_l:

$$\nu(t) = \nu_{l-1}(t) + [\nu_l(t) - \nu_{l-1}(t)]\, e_{01}(\ln\zeta(t); \ln\gamma_l, \ln\delta_l), \qquad t \in \tilde{\Pi}(l), \qquad l = 2, 3, \ldots \tag{7_2}$$

Finally, we set

$$\nu(t) = \nu_1(t) e_{01}(\ln\zeta(t); \ln\gamma_1, \ln\delta_1), \qquad t \in T \cup \tilde{\Pi}(1). \tag{7_3}$$

2. CONSTRUCTION OF THE LOWER CHARACTERISTIC SET OF THE SOLUTION x

2.1. Construction of the Lower Characteristic Set of the Function ϕ

Let us show that the set

$$P \equiv \left\{p \in R^2_- :\ p_1 p_2 = 1,\ \alpha_1 \leq p_1 \leq \alpha_2\right\}$$

is the lower characteristic set of the function ϕ. Obviously, the inequalities

$$2\sqrt{t_1 t_2} + t_2/\alpha_1 \geq -\sqrt{2} t_2/\alpha_1 + t_2/\alpha_1 = -\left(\sqrt{2} - 1\right) t_2/\alpha_1 \geq 0,$$
$$2\sqrt{t_1 t_2} + \alpha_2 t_1 \geq -\sqrt{2}\alpha_2 t_1 + \alpha_2 t_1 = -\alpha_2\left(\sqrt{2} - 1\right) t_1 \geq 0$$

are valid for $\alpha_2^2/2 \leq \theta_t \leq 2\alpha_1^2$. We take an arbitrary vector $p \in P$ and set $R_\phi(p,t) \equiv \ln\phi(t) - (p,t)$; then we estimate the quantity $R_\phi(p,t)$ from below. For $\alpha_2^2/2 \leq \theta_t \leq 2\alpha_1^2$, we obtain the inequality

$$R_\phi(p,t) \geq -2\sqrt{t_1 t_2} - p_1 t_1 - t_2/p_1 = -t_1\left(p_1 + \sqrt{\theta_t}\right)^2 \Big/ p_1 \geq 0. \tag{8_1}$$

Relation (4) with $\theta_t \geq 2\alpha_1^2$ implies the estimate

$$R_\phi(p,t) = t_2/\alpha_1 - p_1 t_1 - t_2/p_1 \geq (1/\alpha_1 - 1/p_1)\, t_2 - p_1 t_1 \geq -p_1 t_1 > 0. \tag{8_2}$$

Finally, by using (4), we obtain the inequality

$$R_\phi(p,t) = \alpha_2 t_1 - p_1 t_1 - t_2/p_1 \geq (\alpha_2 - p_1)\, t_1 \geq 0 \tag{8_3}$$

for $\theta_t \leq \alpha_2^2/2$. The estimates ($8_1$)–($8_3$) imply the inequality

$$\underline{l}_\phi(p) \equiv \varliminf_{t\to\infty} R_\phi(p,t)/\|t\| \geq 0.$$

The relation $\underline{l}_\phi(p) = 0$ and the second condition $\underline{l}_\phi(p + \varepsilon e_i) < 0$, $\varepsilon > 0$, $i = 1, 2$, in the definition [3] of the lower characteristic vector are established in the direction $t_2 = p_1^2 t_1$, $t_1 \to +\infty$. We have thereby justified the inclusion $P \subset P_\phi$. On the other hand, for an arbitrary lower characteristic vector $p = (p_1, p_2) \in P_\phi$, in the directions $t_2 = e$, $t_1 \to +\infty$, and $t_1 = e$, $t_2 \to +\infty$, we obtain the inequalities $p_1 \leq \alpha_2$ and $p_2 \leq 1/\alpha_1$, respectively. Since the set P_ϕ can be represented [3] by a strictly monotone decreasing curve, we find that it necessarily coincides with P.

2.2. PROOF OF THE COINCIDENCE OF THE LOWER CHARACTERISTIC SET OF THE SOLUTION x WITH THE SET P

Let us show that the lower characteristic set P_x of the solution x coincides with the lower characteristic set P_ϕ of the function ϕ. To this end, we prove the existence of the limit

$$\lim_{t\to\infty} \|t\|^{-1}\nu(t)\ln\psi(t) = 0. \tag{9}$$

By setting $\Pi L(l) = \tilde{\Pi}(l)\cup\Pi(l)\cup\tilde{\Pi}(l+1)$, $l\in N$, and by taking some $l\in N$, we estimate the function ν_l in the strip $\Pi L(l)$. Note first that since θ_t belongs to the interval

$$\left[\left(\Delta_j - \tau_j\Big/\left(2\sqrt{l}\,\right)\right)^2, \left(\Delta_{j-1} + \tau_j\Big/\left(2\sqrt{l}\,\right)\right)^2\right],$$

we have the inequalities $\Delta_{j-1} < \Delta_{j-1} + \tau_j/(2\sqrt{l}\,) \le -\sqrt{\theta_t} \le \Delta_j - \tau_j/(2\sqrt{l}\,) < \Delta_j$, which imply that $\left|P_l^{(j)}\left(-\sqrt{\theta_t}\,\right)\right| \le \mu_l$. Since $\zeta(t) \ge \mu_l^4\left(1+\alpha_1^2\right)$ in the strip $\Pi L(l)$, it follows from (6_1) that

$$\begin{aligned}|\nu_l(t)| = \left|P_l^{(j)}\left(-\sqrt{\theta_t}\,\right)\right| &\le \mu_l \le \sqrt[4]{t_1},\\ \theta_t &\in \left[\left(\Delta_j - \tau_j\Big/\left(2\sqrt{l}\,\right)\right)^2, \left(\Delta_{j-1} + \tau_j\Big/\left(2\sqrt{l}\,\right)\right)^2\right],\\ j &= 1,\ldots,k+1, \qquad t\in \Pi l(l).\end{aligned} \tag{10_1}$$

By using the inequality $\zeta(t) \ge \left(\max_{j=0,\ldots,k+1}|g(\Delta_j)|\right)^4\left(1+2\alpha_1^2\right)$, which is valid in each strip $\Pi L(l)$, $l\in N$, from relation (6_2), we obtain the estimates

$$\begin{aligned}|\nu_l(t)| = |g(\Delta_j)| &\le \sqrt[4]{t_1},\\ \theta_t &\in \left[\left(\Delta_j + \tau_{j+1}\Big/\left(4\sqrt{l}\,\right)\right)^2, \left(\Delta_j - \tau_j\Big/\left(4\sqrt{l}\,\right)\right)^2\right],\\ j &= 1,\ldots,k, \qquad t\in \Pi L(l).\end{aligned} \tag{10_2}$$

By virtue of (6_5) and (6_6), we have the inequalities

$$|\nu_l(t)| = |g(\alpha_1)| \le \sqrt[4]{t_1}, \qquad 2\alpha_1^2 \ge \theta_t \ge \left(\alpha_1 + \tau_1\Big/\left(4\sqrt{l}\,\right)\right)^2, \qquad t\in\Pi L(l), \tag{10_3}$$

$$|\nu_l(t)| = |g(\alpha_2)| \le \sqrt[4]{t_1}, \qquad \theta_t \le \left(\alpha_2 - \tau_{k+1}\Big/\left(4\sqrt{l}\,\right)\right)^2, \qquad t\in\Pi L(l). \tag{10_4}$$

Now let

$$\theta_t \in \left(\left(\Delta_j - \tau_j\Big/\left(4\sqrt{l}\,\right)\right)^2, \left(\Delta_j - \tau_j\Big/\left(2\sqrt{l}\,\right)\right)^2\right)$$

with some $j\in\{1,\ldots,k+1\}$ and $t\in\Pi L(l)$. If $g(\Delta_j) - P_l^{(j)}\left(-\sqrt{\theta_t}\,\right) \ge 0$, then from (6_3), we obtain the estimates

$$\begin{aligned}\nu_l(t) &\ge P_l^{(j)}\left(-\sqrt{\theta_t}\,\right) \ge -\left|P_l^{(j)}\left(-\sqrt{\theta_t}\,\right)\right| \ge -\sqrt[4]{t_1},\\ \nu_l(t) &\le P_l^{(j)}\left(-\sqrt{\theta_t}\,\right) + g(\Delta_j) - P_l^{(j)}\left(-\sqrt{\theta_t}\,\right) \le |g(\Delta_j)| \le \sqrt[4]{t_1}.\end{aligned}$$

But if $g(\Delta_j) - P_l^{(j)}\left(-\sqrt{\theta_t}\,\right) < 0$, then we have the inequalities

$$\begin{aligned}\nu_l(t) &\le P_l^{(j)}\left(-\sqrt{\theta_t}\,\right) \le \sqrt[4]{t_1},\\ \nu_l(t) &\ge P_l^{(j)}\left(-\sqrt{\theta_t}\,\right) + g(\Delta_j) - P_l^{(j)}\left(-\sqrt{\theta_t}\,\right) \ge -|g(\Delta_j)| \ge -\sqrt[4]{t_1}.\end{aligned}$$

We have thereby proved the estimate

$$|\nu_l(t)| \le \sqrt[4]{t_1}, \qquad \theta_t \in \left(\left(\Delta_j - \tau_j \Big/ \left(4\sqrt{l}\,\right)\right)^2, \left(\Delta_j - \tau_j \Big/ \left(2\sqrt{l}\,\right)\right)^2\right), \\ j = 1, \ldots, k+1, \qquad t \in \Pi L(l). \tag{10_5}$$

In a similar way, one can show that

$$|\nu_l(t)| \le \sqrt[4]{t_1}, \qquad \theta_t \in \left(\left(\Delta_j + \tau_{j+1} \Big/ \left(2\sqrt{l}\,\right)\right)^2, \left(\Delta_j + \tau_{j+1} \Big/ \left(4\sqrt{l}\,\right)\right)^2\right), \\ j = 0, \ldots, k, \qquad t \in \Pi L(l). \tag{10_6}$$

Therefore, for each $l \in N$ from the estimates (10_1)–(10_6), we obtain the inequality

$$|\nu_l(t)| \le \sqrt[4]{t_1}, \qquad t \in \Pi L(l), \qquad \theta_t \le 2\alpha_1^2. \tag{11}$$

By virtue of definitions (7_1)–(7_3) of the function $\nu(t)$ and inequality (11), we have the estimate

$$|\nu(t)| \le \sqrt[4]{t_1}, \qquad t \in R^2_{>1}, \qquad \theta_t \le 2\alpha_1^2. \tag{12}$$

By using relation (5) and the estimate (12), we obtain the inequality

$$|\nu(t) \ln \psi(t)| \le \sqrt[4]{t_1} \ln t_1, \qquad t \in R^2_{>1}, \tag{13}$$

which implies (9). We have thereby shown that the lower characteristic set P_x of the solution x of Eq. (1_1) coincides with P.

3. PROOF OF RELATION (2) FOR THE SOLUTION x

We take an arbitrary interior point $p = (p_1, \varphi(p_1))$, $\alpha_1 < p_1 < \alpha_2$, of the lower characteristic set P_x. Let the limit $\underline{\ln}_x(p, 0)$ be realized along a sequence $\{t(m)\}$ [6] for which there exists a finite limit $\lim_{m\to\infty} \theta_{t(m)} = \theta > 0$. There are only two possible cases: $\theta \ne p_1^2$ and $\theta = p_1^2$.

Let $\theta \ne p_1^2$. Since $\lim_{m\to\infty} \left(p_1 + \sqrt{\theta_{t(m)}}\,\right)^2 = \left(p_1 + \sqrt{\theta}\,\right)^2 > 0$, without loss of generality, one can assume that the sequence $\{t(m)\}$ satisfies the inequality

$$\left(p_1 + \sqrt{\theta_{t(m)}}\,\right)^2 \ge \left(p_1 + \sqrt{\theta}\,\right)^2 \Big/ 2 > 0, \qquad m \in N.$$

It follows from (8_1)–(8_3) that

$$R_\phi(p,t) \ge M t_1(m), \qquad M = \min\left\{ -\left(p_1 + \sqrt{\theta}\,\right)^2 \Big/ (2p_1)\,; -p_1; \alpha_2 - p_1 \right\} > 0.$$

Then, by using (3), we obtain the estimate

$$\underline{\ln}_x(p,0) \ge \lim_{m\to\infty} \left(M t_1(m) - \sqrt[4]{t_1(m)} \ln t_1(m) \right) / \|\ln t(m)\| = +\infty.$$

Now we suppose that $\theta = p_1^2$. Without loss of generality, we assume that all elements of the sequence $t(m)$ realizing the lower limit $\underline{\ln}_x(p,0)$ belong to the strips $\tilde{\Pi}(l_m) \cup \Pi(l_m)$ with distinct indices $l_m > 1$, $l_{m+1} > l_m \to +\infty$ as $m \to +\infty$. The following two cases are possible: (i) either p_1 does not coincide with any point of discontinuity of the function $g(\Delta)$, i.e., $p_1 \in (\Delta_{j_0-1}, \Delta_{j_0})$, $j_0 \in \{1, \ldots, k+1\}$, or (ii) p_1 coincides with some point of discontinuity Δ_{j_0}, $j_0 \in \{1, \ldots, k\}$, of the function $g(\Delta)$.

By virtue of the inclusion

$$\lim_{m\to\infty}\theta_{t(m)} = p_1^2 \in \left(\left(\Delta_{j_0} - \tau_{j_0}\Big/\left(2\sqrt{l_0}\,\right)\right)^2, \left(\Delta_{j_0-1} + \tau_{j_0}\Big/\left(2\sqrt{l_0}\,\right)\right)^2\right)$$

with some $l_0 \in N$, in the first case without loss of generality, we assume that the sequence $\{t(m)\}$ satisfies the inequalities $\left(\Delta_{j_0} - \tau_{j_0}/\left(2\sqrt{l_0}\,\right)\right)^2 < \theta_{t(m)} < \left(\Delta_{j_0-1} + \tau_{j_0}/\left(2\sqrt{l_0}\,\right)\right)^2$ for all $m \in N$. Again, without loss of generality, we assume that $l_m - 1 \ge l_0$ for all $m \in N$. Consequently,

$$\left(\Delta_{j_0} - \tau_{j_0}\Big/\left(2\sqrt{l_m - 1}\,\right)\right)^2 < \theta_{t(m)} < \left(\Delta_{j_0-1} + \tau_{j_0}\Big/\left(2\sqrt{l_m - 1}\,\right)\right)^2, \qquad \forall m \in N. \tag{14}$$

Without loss of generality, one can restrict considerations to the following two possibilities:

(1) $t(m) \in \Pi\,(l_m)$ for all $m \in N$;

(2) $t(m) \in \tilde{\Pi}\,(l_m)$ for all $m \in N$ as well.

In the first possible case, by using (7_1), (6_1), and inequality (14), we obtain

$$\nu(t(m)) = P_{l_m}^{(j_0)}\left(-\sqrt{\theta_{t(m)}}\,\right),$$

and relations (5) and (14) imply that $\ln\phi(t(m)) = \ln t_1(m)$. Since the polynomials $P_{l_m}^{(j_0)}(\Delta)$ uniformly converge to the function $g_{j_0}(\Delta)$ as $m \to \infty$ on the interval $[\Delta_{j_0-1}, \Delta_{j_0}]$, $\Delta_{j_0-1} < -\sqrt{\theta_{t(m)}} < \Delta_{j_0}$ for all $m \in N$ and since g_{j_0} is a continuous function on the interval $[\Delta_{j_0-1}, \Delta_{j_0}]$, it follows from (8_1)–(8_3) that

$$\begin{aligned}\underline{\ln}_x(p,0) &\ge \lim_{m\to\infty}\left(P_{l_m}^{(j_0)}\left(-\sqrt{\theta_{t(m)}}\,\right)\ln t_1(m)\right)\Big/\|\ln t(m)\| \\ &= \lim_{m\to\infty} P_{l_m}^{(j_0)}\left(-\sqrt{\theta_{t(m)}}\,\right)/\sqrt{2} = g\,(p_1)\Big/\sqrt{2}.\end{aligned}$$

But if the second possibility takes place, then, without loss of generality, we consider only two cases: either $\nu_{l_m}(t(m)) \le \nu_{l_m-1}(t(m))$ for all $m \in N$, or $\nu_{l_m}(t(m)) > \nu_{l_m-1}(t(m))$ for all $m \in N$. In the first case, from (7_2), we obtain the estimate $\nu(t(m)) \ge \nu_{l_m}(t(m))$, whence it follows that $\underline{\ln}_x(p,0) \ge g\,(p_1)\,/\sqrt{2}$. In the second case, from (7_2) and (14), we have

$$\nu(t(m)) \ge \nu_{l_m-1}(t(m)) = P_{l_m-1}^{(j_0)}\left(-\sqrt{\theta_{t(m)}}\,\right).$$

Therefore, we have again obtained the estimates

$$\underline{\ln}_x(p,0) \ge \lim_{m\to\infty}\left(P_{l_m-1}^{(j_0)}\left(-\sqrt{\theta_{t(m)}}\,\right)\ln t_1(m)\right)\Big/\|\ln t(m)\| = g\,(p_1)\,/\sqrt{2}.$$

By virtue of the inequalities $(\Delta_{j_0} - \tau_{j_0}/(2\sqrt{m}\,))^2 < \theta_{\tau(m)} < (\Delta_{j_0-1} + \tau_{j_0}/(2\sqrt{m}\,))^2$, $m \ge l_0$, $m \in N$, along some sequence $\{\tau(m)\}$, $\tau(m) \in \Pi(m)$, $\theta_{\tau(m)} = p_1^2$, $m \ge l_0$, $m \in N$, relations (7_1) and (6_1) imply that $\nu(\tau(m)) = \nu_m(\tau(m)) = P_m^{(j_0)}\left(-\sqrt{\theta_{\tau(m)}}\,\right)$, and consequently,

$$\begin{aligned}&\lim_{m\to\infty}\left(R_\phi(p,\tau(m)) + P_m^{(j_0)}\left(-\sqrt{\theta_{\tau(m)}}\,\right)\ln\tau_1(m)\right)\Big/\|\ln\tau(m)\| \\ &\qquad = \lim_{m\to\infty} P_m^{(j_0)}\left(-\sqrt{\theta_{t(m)}}\,\right)\Big/\sqrt{2} = g\,(p_1)/\sqrt{2}.\end{aligned}$$

We have thereby proved the desired inequality (2) for all points p_1 of continuity of the function g.

Now let p_1 coincide with some point of discontinuity Δ_{j_0}, $j_0 \in \{1,\ldots,k\}$, of the function g. Without loss of generality, we consider the following three possible cases:

(1) $\left(\Delta_{j_0} + \tau_{j_0+1}/\left(4\sqrt{l_m}\,\right)\right)^2 \le \theta_{t(m)} \le \left(\Delta_{j_0} - \tau_{j_0}/\left(4\sqrt{l_m}\,\right)\right)^2$ for all $m \in N$;

(2) $\alpha_1^2 > \theta_{t(m)} > \left(\Delta_{j_0} - \tau_{j_0}/\left(4\sqrt{l_m}\,\right)\right)^2$ for all $m \in N$;

(3) $\alpha_2^2 < \theta_{t(m)} < \left(\Delta_{j_0} + \tau_{j_0+1}/\left(4\sqrt{l_m}\,\right)\right)^2$ for all $m \in N$.

In the first case, we have $\nu_{l_m}(t(m)) = g\left(\Delta_{j_0}\right)$, $m \in N$, and the inequalities

$$\left(\Delta_{j_0} + \tau_{j_0+1}\Big/\left(4\sqrt{l_m - 1}\,\right)\right)^2 \le \theta_{t(m)} \le \left(\Delta_{j_0} - \tau_{j_0}\Big/\left(4\sqrt{l_m - 1}\,\right)\right)^2, \qquad m \in N,$$

and (6_2) imply that

$$\nu_{l_m-1}(t(m)) = g\left(\Delta_{j_0}\right), \qquad m \in N.$$

It follows from (7_1) and (7_2) that $\nu(t(m)) = g\left(\Delta_{j_0}\right)$, $m \in N$. Therefore, we obtain the estimates

$$\underline{\ln}_x(p, 0) \ge \lim_{m\to\infty} \left(g\left(\Delta_{j_0}\right) \ln t_1(m)\right)/\|\ln t(m)\| = g\left(\Delta_{j_0}\right)/\sqrt{2}.$$

If the second possibility takes place, then

$$\sqrt{\theta_{t(m)}} > -\Delta_{j_0} + \tau_{j_0}\Big/\left(4\sqrt{l_m}\,\right), \qquad m \in N.$$

Since $t(m) \in \tilde{\Pi}\left(l_m\right) \cup \Pi\left(l_m\right)$, it follows from the definition of the strips that $\zeta(t(m)) \ge l_m^2\left(1 + \alpha_1^2\right)$, $m \in N$, whence $\sqrt{t_1(m)} \ge l_m$. By using (8_1), we obtain the estimates

$$\begin{aligned} R_\phi(p, t(m)) &\ge -t_1(m)\left(\Delta_{j_0} + \sqrt{\theta_{t(m)}}\,\right)^2\Big/\Delta_{j_0} \ge -t_1(m)\tau_{j_0}^2/\left(16 l_m \Delta_{j_0}\right) \\ &\ge -\sqrt{t_1(m)}\tau_{j_0}^2/\left(16\Delta_{j_0}\right). \end{aligned}$$

Consequently, by virtue of (13), we have

$$\underline{\ln}_x(p, 0) \ge \lim_{m\to\infty} \left(-\sqrt{t_1(m)}\tau_{j_0}^2/\left(16\Delta_{j_0}\right) - \sqrt[4]{t_1(m)} \ln t_1(m)\right)\Big/\|\ln t(m)\| = +\infty.$$

For the third possibility, we have

$$\sqrt{\theta_{t(m)}} < -\left(\Delta_{j_0} + \tau_{j_0+1}/\left(4\sqrt{l_m}\,\right)\right), \qquad m \in N.$$

Since $t(m) \in \tilde{\Pi}\left(l_m\right) \cup \Pi\left(l_m\right)$, we again have the inequality $\sqrt{t_1(m)} \ge l_m$. By using inequality (8_1), we obtain the estimates

$$\begin{aligned} R_\phi(p, t(m)) &\ge -t_1(m)\left(\Delta_{j_0} + \sqrt{\theta_{t(m)}}\,\right)^2\Big/\Delta_{j_0} \ge -t_1(m)\tau_{j_0+1}^2/\left(16 l_m \Delta_{j_0}\right) \\ &\ge -\sqrt{t_1(m)}\tau_{j_0+1}^2/\left(16\Delta_{j_0}\right), \end{aligned}$$

whence it follows that

$$\underline{\ln}_x(p, 0) \ge \lim_{m\to\infty} \left(-\sqrt{t_1(m)}\tau_{j_0+1}^2/\left(16\Delta_{j_0}\right) - \sqrt[4]{t_1(m)} \ln t_1(m)\right)\Big/\|\ln t(m)\| = +\infty.$$

In addition, the inequality $\nu(\tau(m)) = g\left(\Delta_{j_0}\right)$ is valid along the sequence $\{\tau(m)\}$, $\tau(m) \in \Pi(m)$, $\theta_{\tau(m)} = \Delta_{j_0}^2$, $m \in N$; consequently,

$$\lim_{m\to\infty} \left(R_\phi(p, \tau(m)) + g\left(\Delta_{j_0}\right) \ln \tau_1(m)\right)/\|\ln \tau(m)\| = g\left(\Delta_{j_0}\right)/\sqrt{2}.$$

This completes the proof of relation (2) for points Δ_j, $j = 1, \ldots, k$, of discontinuity of the function g as well.

4. CONSTRUCTION OF THE EQUATION. BOUNDEDNESS OF THE COEFFICIENTS

The function $x > 0$ given by (3) is a solution of Eq. (1_1) with the coefficients $a_i(t) = \partial \ln x(t)/\partial t_i$, $t \in R^2_{>1}$, $i = 1, 2$, satisfying the condition of the complete integrability by virtue of the infinite differentiability of $\ln x$ in $R^2_{>1}$.

By using the inequality

$$0 \le \frac{de_{01}(\tau; \tau_1, \tau_2)}{d\tau} \le \begin{cases} 2\exp\left[2(\tau_2 - \tau_1)^{-2}\right] & \text{if} \quad \tau_2 - \tau_1 \le 1/2 \\ 4 & \text{if} \quad \tau_2 - \tau_1 \ge 2, \end{cases} \tag{15}$$

from the lemma in [11], we show that these coefficients are bounded.

First, let us show that the derivatives $\partial \ln \phi(t)/\partial t_i$, $i = 1, 2$, are bounded. If $\alpha_2^2/2 \le \theta_t \le 2\alpha_1^2$, then we have the estimates

$$|\partial \ln \phi(t)/\partial t_1| \le 3\sqrt{\theta_t} - \alpha_2 + e^8\left(2\theta_t^{3/2} - \theta_t^2/\alpha_1\right)\Big/\alpha_1^2 + 2e^8\left(2\theta_t^{3/2} - \alpha_2\theta_t\right)\Big/\alpha_2^2 \le \sigma_1,$$

$$|\partial \ln \phi(t)/\partial t_2| \le 3\sqrt{\theta_t^{-1}} - 1/\alpha_1 + e^8\left(2\sqrt{\theta_t} - \theta_t/\alpha_1\right)\Big/\alpha_1^2 + 2e^8\left(2\sqrt{\theta_t} - \alpha_2\right)\Big/\alpha_2^2 \le \sigma_1$$

with some constant $\sigma_1 > 0$. The fact that the same derivatives are bounded for $\theta_t < \alpha_2^2/2$ and $\theta_t > 2\alpha_1^2$ is obvious.

Let us now proceed to proving the boundedness of partial derivatives of the product $\nu(t)\ln\psi(t)$. By (5), we have

$$\nu(t)\ln\psi(t) = 0, \qquad t \in R^2_{>1}, \qquad \theta_t \in \left(0, \alpha_2^2/2\right] \cup \left[2\alpha_1^2, +\infty\right), \tag{16}$$

which implies that it suffices to prove the boundedness of the derivatives $\partial(\nu(t)\ln\psi(t))/\partial t_i$, $i = 1, 2$, for $\theta_t \in (\alpha_2^2/2, 2\alpha_1^2)$. By using (12) and (5), we obtain

$$|\nu(t)\partial \ln\psi(t)/\partial t_1| \le \sqrt[4]{t_1}\left(2e^8\left(2\alpha_1^2/\alpha_2^2 + 1\right)(\ln t_1)/t_1 + 1/t_1\right) \le \sigma_2,$$
$$t \in R^2_{>1}, \qquad \theta_t \in \left(\alpha_2^2/2, 2\alpha_1^2\right),$$
$$|\nu(t)\partial \ln\psi(t)/\partial t_2| \le \sqrt[4]{t_1}\left(2e^8/\left(t_1\alpha_2^2\right) + e^8/\left(t_1\alpha_1^2\right)\right)\ln t_1 \le \sigma_2,$$
$$t \in R^2_{>1}, \qquad \theta_t \in \left(\alpha_2^2/2, 2\alpha_1^2\right),$$

with some constant $\sigma_2 > 0$.

We take some $l \in N$ and estimate the derivatives of the function ν_l in the strip $\Pi L(l)$. Let

$$\theta_t \in \left[\left(\Delta_j - \tau_j\Big/\left(2\sqrt{l}\,\right)\right)^2, \left(\Delta_{j-1} + \tau_j\Big/\left(2\sqrt{l}\,\right)\right)^2\right]$$

with some $j \in \{1, \ldots, k+1\}$, $t \in \Pi L(l)$. Then from (6_1) and from the inequality $\zeta(t) \ge \varrho_l^2\left(1 + \alpha_1^2\right)$, $t \in \Pi L(l)$, we obtain

$$|\partial\nu_l(t)/\partial t_1| = \left|dP_l^{(j)}(\Delta)/d\Delta\right|_{\Delta = -\sqrt{\theta_t}} \sqrt{\theta_t}/(2t_1) \le -\varrho_l\alpha_1/(2t_1) \le -\alpha_1/\left(2\sqrt{t_1}\,\right), \tag{17_1}$$

$$|\partial\nu_l(t)/\partial t_2| \le \varrho_l\Big/\left(2\sqrt{\theta_t}t_1\right) \le -1/\left(2\alpha_2\sqrt{t_1}\,\right). \tag{17_2}$$

But if either

$$\theta_t \in \left[\left(\Delta_j + \tau_{j+1}\Big/\left(4\sqrt{l}\,\right)\right)^2, \left(\Delta_j - \tau_j\Big/\left(4\sqrt{l}\,\right)\right)^2\right], \qquad j \in \{1, \ldots, k\},$$

or $\theta_t \ge \left(\alpha_1 + \tau_1\Big/\left(4\sqrt{l}\,\right)\right)^2$ or $\theta_t \le \left(\alpha_2 - \tau_{k+1}\Big/\left(4\sqrt{l}\,\right)\right)^2$, then

$$\partial\nu_l(t)/\partial t_i = 0, \qquad i = 1, 2. \tag{17_3}$$

If $\theta_t \in \left(\left(\Delta_j - \tau_j\Big/\left(4\sqrt{l}\,\right)\right)^2, \left(\Delta_j - \tau_j\Big/\left(2\sqrt{l}\,\right)\right)^2\right)$, $j \in \{1, \ldots, k+1\}$, $t \in \Pi L(l)$, then, by analogy with (17_1) and (17_2), from (6_3) and the inequalities $t_1 \geq \mu_l^4 l^2$ and $t_1 \geq |g(\Delta_j)|^4 l^2$, we obtain the estimates

$$\begin{aligned} |\partial\nu_l(t)/\partial t_1| &\leq -\alpha_1/\sqrt{t_1} + 2e^8\left(|g(\Delta_j)| + \mu_l\right)\sqrt{l}\sqrt{\theta_t}/(t_1\tau_j) \\ &\leq -\alpha_1/\sqrt{t_1} - 4e^8\alpha_1\Big/\left(\sqrt[4]{t_1^3}\tau_j\right), \end{aligned} \tag{17_4}$$

$$\begin{aligned} |\partial\nu_l(t)/\partial t_2| &\leq -1/\left(\alpha_2\sqrt{t_1}\,\right) + 2e^8\left(|g(\Delta_j)| + \mu_l\right)\sqrt{l}\Big/\left(\sqrt{\theta_t}t_1\tau_j\right) \\ &\leq -1/\left(\alpha_2\sqrt{t_1}\,\right) - 4e^8\Big/\left(\alpha_2\sqrt[4]{t_1^3}\tau_j\right). \end{aligned} \tag{17_5}$$

Finally, if $\theta_t \in \left(\left(\Delta_j + \tau_{j+1}\Big/\left(2\sqrt{l}\,\right)\right)^2, \left(\Delta_j + \tau_{j+1}\Big/\left(4\sqrt{l}\,\right)\right)^2\right)$, $j \in \{0, \ldots, k\}$, $t \in \Pi L(l)$, then, in a similar way, it follows from (6_4) that

$$\begin{aligned} |\partial\nu_l(t)/\partial t_1| &\leq -\alpha_1/\left(2\sqrt{t_1}\,\right) + 2e^8\left(\mu_l + |g(\Delta_j)|\right)\sqrt{l}\sqrt{\theta_t}/(t_1\tau_{j+1}) \\ &\leq -\alpha_1/\left(2\sqrt{t_1}\,\right) - 4e^8\alpha_1\Big/\left(\sqrt[4]{t_1^3}\tau_{j+1}\right), \end{aligned} \tag{17_6}$$

$$\begin{aligned} |\partial\nu_l(t)/\partial t_2| &\leq -1/\left(2\alpha_2\sqrt{t_1}\,\right) + 2e^8\left(\mu_l + |g(\Delta_j)|\right)\sqrt{l}\Big/\left(\sqrt{\theta_t}t_1\tau_{j+1}\right) \\ &\leq -1/\left(2\alpha_2\sqrt{t_1}\,\right) - 4e^8\Big/\left(\alpha_2\sqrt[4]{t_1^3}\tau_{j+1}\right). \end{aligned} \tag{17_7}$$

The estimates (17_1)–(17_7) imply the inequality

$$|\partial\nu_l(t)/\partial t_i| \leq \sigma_3/\sqrt{t_1}, \qquad t \in \Pi L(l), \qquad i = 1, 2, \tag{18}$$

with some constant $\sigma_3 > 0$.

From definition (7_1) of the function $\nu(t)$ in the "basic" strips $\Pi(l)$ and from (18), we obtain the estimates

$$|\partial\nu(t)/\partial t_i| \leq \sigma_3/\sqrt{t_1}, \qquad t \in \Pi(l), \qquad l \in N, \qquad i = 1, 2. \tag{19_1}$$

By using (7_2), (7_3), (18), (11), and the second inequality in (15), we obtain the estimates

$$\begin{gathered} |\partial\nu(t)/\partial t_i| \leq \sigma_3/\sqrt{t_1} + 8\sqrt[4]{t_1}/\zeta(t) \leq \sigma_3/\sqrt{t_1} + 8/\sqrt[4]{t_1^3}, \\ t \in \tilde{\Pi}(l), \qquad l \in N, \qquad \theta_t \leq 2\alpha_1^2, \qquad i = 1, 2. \end{gathered} \tag{19_2}$$

Therefore, it follows from inequalities (19_1), (19_2), the relation $\nu(t) = 0$, $t \in T$, the estimate $|\ln\psi(t)| \leq \ln t_1$, $t \in R^2_{>1}$, and from (16) that the products $(\partial\nu(t)/\partial t_i)\ln\psi(t)$, $t \in R^2_{>1}$, $i = 1, 2$, are bounded. We have thereby shown that the coefficients of system (1_1) are bounded. The proof of Theorem 1 for $\alpha_1 < \alpha_2 < 0$ is complete.

But if $\alpha_2 \geq 0$, then on the basis of the function g, we introduce a new function

$$g_1 : [\alpha_1 - \alpha_2 - 1, -1] \to R$$

by setting $g_1(\Delta) = g(\Delta + \alpha_2 + 1)$. Then, just as above, for the function g_1, we construct an infinitely differentiable function $x_1 > 0$ so as to ensure that this function has bounded derivatives $\partial\ln x_1(t)/\partial t_i$, $i = 1, 2$, the domain of the curve of its lower characteristic set P_{x_1} coincides with the interval $[\alpha_1 - \alpha_2 - 1, -1]$, and the function itself satisfies the relation $\underline{c}_{x_1}(p_1[x_1]) = g_1(p_1[x_1])$ for all $p_1[x_1] \in (\alpha_1 - \alpha_2 - 1, -1)$. The lower characteristic vectors $(p_1[x], \varphi(p_1[x])) \in P_x$ and $(p_1[x_1], \varphi_1(p_1[x_1])) \in P_{x_1}$ of the functions $x(t) = x_1(t)\exp[(\alpha_2 + 1)t_1]$ and $x_1(t)$, respectively, are related by the conditions $p_1[x] = p_1[x_1] + \alpha_2 + 1 \in [\alpha_1, \alpha_2]$ and $\varphi(p_1[x]) = \varphi_1(p_1[x] - \alpha_2 - 1)$, and the lower characteristic degree functions are related by the formulas

$$\underline{c}_x(p_1[x]) = \underline{c}_{x_1}(p_1[x_1]) = g_1(p_1[x_1]) = g(p_1[x]).$$

Obviously, the function x is a solution of a completely integrable Pfaff equation (1_1) with bounded infinitely differentiable coefficients, which completes the proof of Theorem 1.

Theorem 2. *For an arbitrary positive integer n and for an arbitrary piecewise continuous function $g : [a_1, a_2] \to R$, there exists a completely integrable Pfaff system (1_n) with infinitely differentiable bounded coefficients such that, for each nontrivial solution $x : R^2_{>1} \to R^n \backslash \{0\}$ of this system, the domain of the curve Λ_x coincides with the interval $[a_1, a_2]$ and $\bar{c}_x(\lambda_1) = g(\lambda_1)$ for all $\lambda_1 \in (a_1, a_2)$.*

Proof. We use the function g to introduce a new piecewise continuous function

$$g_1 : [-a_2, -a_1] \to R$$

by setting $g_1(\Delta) = -g(-\Delta)$. Following Theorem 1, we construct an infinitely differentiable function $x_1 > 0$ such that the function has bounded derivatives $\partial \ln x_1(t)/\partial t_i$, $i = 1, 2$, the domain of the curve P_{x_1} of this function coincides with the interval $[-\alpha_2, -\alpha_1]$, and $\underline{c}_{x_1}(p_1[x_1]) = g_1(p_1[x_1])$ for all $p_1[x_1] \in (-\alpha_2, -\alpha_1)$. We use the function x_1 to define a solution $x = x_1^{-1}$ of the completely integrable Pfaff equation (1_1) with bounded infinitely differentiable coefficients $a_i(t) = \partial \ln x(t)/\partial t_i$, $t \in R^2_{>1}$, $i = 1, 2$. Note that the characteristic vector $\lambda[x]$ of the solution x is equal to the lower characteristic vector $p[x_1]$ of the function x_1 with the opposite sign, i.e., $\lambda[x] = -p[x_1]$. Therefore,

$$\begin{aligned}\bar{c}_x(\lambda_1[x]) &= \sqrt{2}\,\overline{\ln}_x(\lambda[x], 0) = -\sqrt{2}\,\underline{\ln}_{x_1}(p[x_1], 0) = -\underline{c}_{x_1}(p_1[x_1]) \\ &= -g_1(p_1[x_1]) = g(-p_1[x_1]) = g(\lambda_1[x])\end{aligned}$$

for all $\lambda_1[x] \in (a_1, a_2)$, which completes the proof of Theorem 2.

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