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ORDINARY DIFFERENTIAL EQUATIONS

Necessary Properties of Boundary Degree Sets of Solutions to Linear Pfaff Systems

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Consider the linear Pfaff system

$$\partial x/\partial t_i = A_i(t)x, \qquad x \in R^n, \qquad t = (t_1, t_2) \in R^2_{\geq 1}, \qquad i = 1, 2, \tag{1}$$

with continuously differentiable matrix functions $A_1(t)$ and $A_2(t)$ bounded in $R^2_{\geq 1}$ and satisfying the complete integrability condition [1, pp. 14–24; 2, pp. 16–26]

$$\partial A_1(t)/\partial t_2 + A_1(t)A_2(t) = \partial A_2(t)/\partial t_1 + A_2(t)A_1(t), \qquad t \in R^2_{\geq 1}.$$

Let $p = p[x]$ be some lower characteristic vector [3] of a nontrivial solution $x : R^2_{\geq 1} \to R^n \backslash \{0\}$ of system (1), and let $P_x = \cup\, p[x]$ be the lower characteristic set of x [3]. This set is determined [3] by a monotone decreasing concave function $\varphi : [\alpha_x, \beta_x] \to [a_x, b_x]$ by the formula $P_x = \{(p_1, \varphi(p_1)) \in R^2 : \ \alpha_x \leq p_1 \leq \beta_x\}$. The notion of the (bounded) lower characteristic degree $d = d_x(p) \in R^2$ of x associated with the lower characteristic vector $p \in P_x$ was defined in [4] by the conditions

$$\underline{\ln}_x(p, d) \equiv \varliminf_{t\to\infty} \frac{\ln\|x(t)\| - (p, t) - (d, \ln t)}{\|\ln t\|} = 0, \qquad \ln t \equiv (\ln t_1, \ln t_2) \in R^2_+, \tag{2_1}$$

$$\underline{\ln}_x(p, d + \varepsilon e_i) < 0, \qquad e_i = (2 - i, i - 1) \in R^2, \qquad \forall \varepsilon > 0, \qquad i = 1, 2. \tag{2_2}$$

Note that, unlike the lower characteristic set P_x, the lower degree set $\underline{D}_x(p) \equiv \cup\, d_x(p)$ of a nontrivial solution of a completely integrable Pfaff system (1) with bounded continuously differentiable coefficients can be empty. For example, the lower degree set of the nontrivial solution $x(t) = (e^{-t_1} + e^{-t_2}) \exp\left\{\ln^2 t_1 + \ln^2 t_2\right\}$ of the Pfaff equation

$$\partial x/\partial t_i = a_i(t)x, \qquad a_i(t) = x^{-1}(t)\partial x(t)/\partial t_i, \qquad t = (t_1, t_2) \in R^2_{\geq 1}, \qquad i = 1, 2, \tag{1_1}$$

corresponding to an arbitrary point of the lower characteristic set is empty.

The lower degree set $\underline{D}_x(p)$ is referred to as an *interior lower degree set* if the point $p = (p_1, \varphi(p_1)) \in P_x$, $p_1 \in (\alpha_x, \beta_x)$, is an interior point of the lower characteristic set P_x and as a *left* (respectively, *right*) *boundary lower degree set* if $p \equiv p' = (\alpha_x, b_x)$ [respectively, $p \equiv p'' = (\beta_x, a_x)$] is a "left" (respectively, "right") boundary point of the lower characteristic set.

An arbitrary nonempty interior lower degree set $\underline{D}_x(p)$ of a nontrivial solution $x(t)$ of system (1) is completely described in [4]. It is a line of the form $d_1 + d_2 = \underline{c}_x(p)$ on the plane R^2. One can readily see that the left boundary lower degree set of the nontrivial solution

$$x(t) = \left(e^{-t_1} + e^{-t_2}\right)\psi(t), \qquad \psi(t) = \ln^2 t_2 e_{01}(t_2/t_1; 2, 3), \qquad t \in R^2_{\geq 1}, \tag{3}$$

of the Pfaff equation (1_1) with bounded infinitely differentiable coefficients, which is constructed with the use of the infinitely differentiable function [5, p. 54 of the Russian translation]

$$e_{01}(\eta; \eta_1, \eta_2) = \begin{cases} 0 & \text{for} \quad \eta \in (-\infty, \eta_1] \\ \exp\left\{-(\eta - \eta_1)^{-2} \exp\left[-(\eta - \eta_2)^{-2}\right]\right\} & \text{for} \quad \eta \in (\eta_1, \eta_2) \\ 1 & \text{for} \quad \eta \in [\eta_2, +\infty), \end{cases}$$

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$-\infty < \eta_1 < \eta_2 < +\infty$, is the line $d_1 + d_2 = 0$ on the two-dimensional plane. Therefore, we naturally encounter the problem as to whether every nonempty boundary lower degree set of a nontrivial solution of the Pfaff system (1) is a line of the form $d_1 + d_2 = \text{const}$ on the two-dimensional plane.

Obviously, the left boundary lower degree set of the nontrivial solution

$$x(t) = e^{-t_1} + e^{-t_2} \tag{4}$$

of the completely integrable Pfaff equation (1_1) with bounded infinitely differentiable coefficients does not coincide with the line $d_1 + d_2 = 0$ on the two-dimensional plane but is only the half-line $\{d = (d_1, d_2) :\ d_1 + d_2 = 0,\ d_2 \le 0\}$.

Moreover, Theorem 1 below shows that a boundary lower degree set does not necessarily coincide with any line on the two-dimensional plane and even does not necessarily contain a segment of a line.

Theorem 1. *There exists a completely integrable Pfaff equation* (1_1) *with infinitely differentiable bounded coefficients such that the left boundary lower degree set of any nontrivial solution* $x : R^2_{\ge 1} \to R\backslash\{0\}$ *of this equation is a monotone decreasing concave curve* Γ (*which is not a straight line and does not contain any straight-line segment*) *with the range* $\{d_1 : (d_1, d_2) \in \Gamma\} = (-\infty, +\infty)$ *of the first and* $\{d_2 : (d_1, d_2) \in \Gamma\} = (-\infty, 0)$ *of the second component.*

To prove this theorem and the forthcoming assertions, we use the following lemma, establishing some properties of sequences realizing boundary lower degree sets.

Lemma 1. *Let the lower characteristic set* P_x *of a nontrivial solution* $x(t) \ne 0$ *of the Pfaff system* (1) *consist of more than one point, and let* p' *and* p'' *be its left and right boundary points, respectively. Then, for each vector* $N \in R^2$, *there exist sequences* $\{t'(k)\} \uparrow \infty$ *and* $\{t''(k)\} \uparrow \infty$ *realizing the limits* $\underline{\ln}_x(p', N)$ *and* $\underline{\ln}_x(p'', N)$ *and satisfying the following conditions:*

(1) $0 \le \lim_{k\to\infty} \ln t'_1(k)/\|\ln t'(k)\| \le 1/\sqrt{2}$ *and* $0 \le \lim_{k\to\infty} \ln t''_2(k)/\|\ln t''(k)\| \le 1/\sqrt{2}$;

(2) $t'_i(k) \to +\infty$ *and* $t''_i(k) \to +\infty$ *as* $k \to +\infty$, $i = 1, 2$.

Proof. Let us prove the desired assertion for the left boundary point $p' \in \partial P_x$. The right boundary point $p'' \in \partial P_x$ can be considered in a similar way. By [4, 6, 7], there exists a sequence $\{t'(k)\} \uparrow \infty$ realizing the limit $\underline{\ln}_x(p', N)$ and such that $t'_1(k)/t'_2(k) \to \gamma' \in [0, +\infty)$ as $k \to +\infty$. Since the norm $\|t'(k)\|$ of an element of this sequence tends to $+\infty$ as $k \to +\infty$, we have $t'_2(k) \to +\infty$ as $k \to +\infty$. Therefore, without loss of generality, we can assume that the sequence $\{t'(k)\}$ satisfies the inequalities $0 \le \lim_{k\to\infty} \ln t'_1(k)/\|\ln t'(k)\| \le 1/\sqrt{2}$. But if the sequence $\{t'_1(k)\}$ has a subsequence $\{t'_1(k_n)\}$ tending to $+\infty$, then $\{t'(k_n)\}$ is the desired sequence.

Now we suppose that the sequence $\{t'_1(k)\}$ has no subsequence convergent to $+\infty$. Then, by the Bolzano–Weierstrass theorem, there exists a convergent subsequence $\{t'_1(k_n)\}$. Obviously, the subsequence $\{t'(k_n)\}$ satisfies conditions (1) and (2) of Lemma 1 for $i = 2$. Without loss of generality, we can assume that $\{t'(k)\}$ itself is a sequence such that $t'_1(k) \to \alpha \in R$, $\alpha \ge 1$, as $k \to +\infty$.

On the basis of the sequence $\{t'(k)\}$, we construct the new sequence $\{\tau'(k)\}$ with elements $\tau'(k) = (t'_1(k) + \ln\ln t'_2(k), t'_2(k))$. Obviously, this sequence satisfies conditions (1) and (2) of Lemma 1. Let us show that it realizes the lower limit $\underline{\ln}_x(p', N)$.

Since $x(t) \ne 0$ is a nontrivial solution of the Pfaff system (1) with bounded coefficients, we have the inequalities [1, p. 91]

$$\exp\{-a_1|t_1 - \tau_1| - a_2|t_2 - \tau_2|\} \le \|x(t)\|/\|x(\tau)\| \le \exp\{a_1|t_1 - \tau_1| + a_2|t_2 - \tau_2|\}, \quad \forall t, \tau \in R^2_+.$$

Setting $t \equiv \tau'(k)$ and $\tau \equiv t'(k)$ in these inequalities, we obtain the estimates

$$\exp\{-a_1 \ln\ln t'_2(k)\}\,\|x(t'(k))\| \le \|x(\tau'(k))\| \le \exp\{a_1 \ln\ln t'_2(k)\}\,\|x(t'(k))\|. \tag{5}$$

Setting $R_x(p, N, t) \equiv \ln\|x(t)\| - (p, t) - (N, \ln t)$ and $R^1_x(p, N, t) \equiv R_x(p, N, t)/\|\ln t\|$ and using the estimates (5), we arrive at the inequalities

$$\begin{aligned} &[-(p'_1 + a_1)\ln\ln t'_2(k) - N_1\ln(1 + (\ln\ln t'_2(k))/t'_1(k)) + \|\ln t'(k)\| R^1_x(p', N, t'(k))]/\|\ln\tau'(k)\| \\ &\quad \le R^1_x(p', N, \tau'(k)) \le [-(p'_1 - a_1)\ln\ln t'_2(k) - N_1\ln(1 + (\ln\ln t'_2(k))/t'_1(k)) \\ &\qquad\qquad + \|\ln t'(k)\| R^1_x(p', N, t'(k))]/\|\ln\tau'(k)\|. \end{aligned} \tag{6}$$

Let us show that

$$\lim_{k\to\infty} \|\ln t'(k)\| \,/\, \|\ln \tau'(k)\| = 1. \tag{7}$$

Since $t_1'(k) \to \alpha \in R$ as $k \to +\infty$, we have $\lim_{k\to\infty} \ln t_1'(k)/\ln t_2'(k) = 0$. Since $\{t_1'(k)\}$ is a convergent sequence, it follows that it is bounded by some constant α_1 and

$$0 \le \ln(\tau_1'(k)) \,/\, \ln t_2'(k) \le \ln(\alpha_1 + \ln\ln t_2'(k)) \,/\, \ln t_2'(k).$$

Since a sequence bounded above and below by two sequences with a common limit converges to the same limit, it follows that $\lim_{k\to\infty} \ln(\tau_1'(k)) \,/\, \ln t_2'(k) = 0$, which completes the proof of (7).

Since the limit $\underline{\ln}_x (p', N)$ is realized on the sequence $\{t'(k)\}$, we have

$$\underline{\ln}_x (p', N) = \lim_{k\to\infty} R_x^1 (p', N, t'(k)).$$

Therefore, passing to the limit in (6), we obtain $\lim_{k\to\infty} R_x^1 (p', N, \tau'(k)) = \underline{\ln}_x (p', N)$, i.e., the lower limit $\underline{\ln}_x (p', N)$ is realized by the sequence $\{\tau'(k)\}$, which satisfies the assumptions of Lemma 1. The proof of the lemma is complete.

Proof of Theorem 1. In the closed quadrant $R^2_{\ge 1}$ of the two-dimensional plane, we construct the desired Pfaff equation (1_1) with infinitely differentiable bounded coefficients $a_1(t)$ and $a_2(t)$ satisfying the total integrability conditions $\partial a_1(t)/\partial t_2 \equiv \partial a_2(t)/\partial t_1$, $t \in R^2_{\ge 1}$, by constructing a nontrivial solution of this equation.

1. THE CONSTRUCTION OF A SOLUTION

We construct the desired solution $x(t)$ in the form $\ln x(t) = \ln\varphi(t) + \ln\psi(t)$, where $\varphi(t)$ is the function given by the formula $\ln\varphi(t) = \ln(e^{-t_1} + e^{-t_2})$. The function $\ln\psi(t)$ (required for the realization of the desired left boundary lower degree set) is constructed on the basis of the function $v(t) = -\sqrt{\ln t_1 \ln(t_2/t_1)}$. To paste various infinitely differentiable functions together with the preservation of this property, we use the infinitely differentiable function [5, p. 54 of the Russian translation] $e_{01}(\eta; \eta_1, \eta_2)$, $-\infty < \eta_1 < \eta_2 < +\infty$. Let us construct the auxiliary function

$$\ln u(t) = \begin{cases} v(t) & \text{if } \ 3t_1 \le t_2 \le t_1^{t_1} \\ v(t)\left[1 - e_{01}\left(\ln t_2/\left(t_1 \ln t_1\right); 1, 2\right)\right] & \text{if } \ \ t_2 > t_1^{t_1} \\ v(t) e_{01}\left(t_2/t_1; 2, 3\right) & \text{if } \ \ t_2 < 3t_1, \end{cases}$$

defined for all $t \in R^2_{>1}$. We define the function $\psi(t)$ by the formula $\ln\psi(t) = \ln u(t) e_{01}(t_1; 2, 3)$ for $t \in R^2_{>1}$ and $\ln\psi(t) = 0$ for $t_1 = 1$.

2. THE CONSTRUCTION OF THE EQUATION. BOUNDEDNESS OF THE COEFFICIENTS

The above-constructed function $x(t) > 0$ is a solution of the Pfaff equation (1_1) with coefficients $a_1(t) = x^{-1}(t)\partial x(t)/\partial t_1 = \partial \ln x(t)/\partial t_1$ and $a_2(t) = x^{-1}(t)\partial x(t)/\partial t_2 = \partial \ln x(t)/\partial t_2$, $t \in R^2_{\ge 1}$, satisfying the complete integrability condition, since $\ln x(t)$ is infinitely differentiable in $R^2_{\ge 1}$.

Let us show that these coefficients are bounded. First, we note that the partial derivatives of the function $v(t)$ for $3t_1 \le t_2 \le t_1^{t_1}$ satisfy the estimates

$$\begin{aligned} \left|\frac{\partial v(t)}{\partial t_1}\right| &= \left|\left(\frac{1}{t_1}\ln\frac{t_2}{t_1} - \frac{1}{t_1}\ln t_1\right) \Big/ (2v(t))\right| \le \frac{1}{2\sqrt{\ln t_1}}\frac{1}{t_1}\sqrt{\ln(t_2/t_1)} + \frac{\sqrt{\ln t_1}}{2t_1\sqrt{\ln(t_2/t_1)}} \\ &\le \frac{\sqrt{t_1 - 1}}{2t_1} + \frac{\sqrt{\ln t_1}}{2t_1\sqrt{\ln 3}} \le \frac{1}{2} + \frac{1}{2\sqrt{\ln 3}}, \\ \left|\frac{\partial v(t)}{\partial t_2}\right| &= \left|\frac{1}{2v(t)}\frac{1}{t_2}\ln t_1\right| = \frac{\sqrt{\ln t_1}}{2t_2\sqrt{\ln(t_2/t_1)}} \le \frac{1}{6\sqrt{\ln 3}}. \end{aligned}$$

Using the inequality [4, p. 902]

$$\left|\frac{\partial e_{01}\left(\eta\left(t_1,t_2\right);\eta_1,\eta_2\right)}{\partial t_i}\right| \le 4\exp\left[3\left(\eta_2-\eta_1\right)^{-2}\right]\left|\frac{\partial\eta}{\partial t_i}\right|,\qquad i=1,2,$$

valid on an arbitrary closed interval $[\eta_1,\eta_2]$ of length $\eta_2-\eta_1\le 1$, we find that the derivatives of the function $\ln\psi(t)$ are bounded if $t_1^{t_1}<t_2<t_1^{2t_1}$:

$$\left|\frac{\partial\ln\psi(t)}{\partial t_1}\right| \le \left(\frac{\sqrt{\ln\left(t_2/t_1\right)}}{2t_1\sqrt{\ln t_1}}+\frac{\sqrt{\ln t_1}}{2t_1\sqrt{\ln\left(t_2/t_1\right)}}\right)\left|1-e_{01}\left(\frac{\ln t_2}{t_1\ln t_1};1,2\right)\right| + 4e^3|v(t)|\left(\ln t_2\right)\frac{1+\ln t_1}{\left(t_1\ln t_1\right)^2}$$
$$\le \frac{\sqrt{2t_1-1}}{2t_1}+\frac{1}{2\sqrt{\ln 3}}+8e^3\sqrt{2t_1-1}\,\frac{1+\ln t_1}{t_1}\le\frac{1}{\sqrt{2}}+\frac{1}{2\sqrt{\ln 3}}+24\sqrt{2}e^3,$$
$$\left|\frac{\partial\ln\psi(t)}{\partial t_2}\right| \le \frac{1}{6\sqrt{\ln 3}}+4e^3\frac{|v(t)|}{t_1\ln t_1}\le\frac{1}{6\sqrt{\ln 3}}+4\sqrt{2}e^3.$$

In a similar way, we can obtain the estimates

$$\left|\frac{\partial\ln\psi(t)}{\partial t_i}\right| \le \frac{1}{\sqrt{\ln 2}}+12e^3\sqrt{\ln 3},\qquad i=1,2,\qquad 2t_1<t_2<3t_1.$$

The boundedness of the remaining derivatives is obvious. This completes the proof of the boundedness of the coefficients of the Pfaff equation thus constructed.

3. EVALUATION OF THE LOWER CHARACTERISTIC SET

Note that the lower characteristic set of the solution $x(t)$ of Eq. (1_1) coincides with the lower characteristic set $P_\varphi\equiv\left\{p\in R_-^2:\ p_1+p_2=-1\right\}$ of the function $\varphi(t)$. This follows from the existence of the limit $\lim_{t\to\infty}\ln\psi(t)/\|t\|=0$, which, in turn, is a consequence of the estimate $0\ge\ln\psi(t)\ge v(t)\ge-\sqrt{\ln t_1\ln t_2}\ge-\ln\|t\|$.

4. EVALUATION OF THE LEFT BOUNDARY LOWER DEGREE SET

We choose the left boundary point $p'=(-1,0)\in\partial P_x$ of the lower characteristic set and show that the left boundary lower degree set $\underline{D}_x(p')$ coincides with the set

$$D\equiv\left\{d\in R^2:\ d_1=\left(2-\alpha_0\right)/\left(2\sqrt{\alpha_0-1}\right),\ d_2=-1/\left(2\sqrt{\alpha_0-1}\right),\ \alpha_0\in(1,+\infty)\right\}.$$

First, we prove the inclusion $D\subset\underline{D}_x(p')$. We choose an arbitrary vector $d\in D$, and as a sequence realizing the lower limit $\underline{\ln}_x(p',d)$ we choose a sequence $\{t(k)\}$ satisfying the assumptions of Lemma 1. Therefore, without loss of generality, we can assume that the sequence $\{t(k)\}$ satisfies the inequality $t_1(k)>1$ for all $k\in N$. Then we obtain the lower bounds

$$R_x\left(p',d,t(k)\right)\ge\ln\left(1+e^{t_1(k)-t_2(k)}\right)+v(t(k))-\frac{2-\alpha_0}{2\sqrt{\alpha_0-1}}\ln t_1(k)+\frac{\alpha(k)}{2\sqrt{\alpha_0-1}}\ln t_1(k)$$
$$\ge-\sqrt{\alpha(k)-1}\ln t_1(k)-\frac{2-\alpha_0}{2\sqrt{\alpha_0-1}}\ln t_1(k)+\frac{\alpha(k)}{2\sqrt{\alpha_0-1}}\ln t_1(k)$$
$$=\left[2\sqrt{\alpha_0-1}-\sqrt{\alpha(k)-1}+\frac{\alpha(k)-1}{2\sqrt{\alpha_0-1}}\right]\ln t_1(k)$$
$$=\frac{\left[\sqrt{\alpha_0-1}-\sqrt{\alpha(k)-1}\right]^2}{2\sqrt{\alpha_0-1}}\ln t_1(k)\ge 0$$

for $\alpha(k)=\ln t_2(k)/\ln t_1(k)\ge 1$. This, together with the form of the function $x(t)$, implies the inequality $\underline{\ln}_x(p',d)\ge 0$. Consider the direction $t_2=t_1^{\alpha_0}$ for sufficiently large t_1 ($t_1>3$,

$3t_1 < t_1^{\alpha_0} < t_1^{t_1}$). Then, in this direction, the function $\ln\psi(t)$ has the form $\ln\psi(t) = v(t) = -\sqrt{\alpha_0 - 1}\ln t_1$. Consequently,

$$\lim_{t_2=t_1^{\alpha_0}\to\infty}\Big[\ln\Big(1+e^{t_1-t_1^{\alpha_0}}\Big) - \sqrt{\alpha_0-1}\ln t_1 - (2-\alpha_0)/\big(2\sqrt{\alpha_0-1}\,\big)\ln t_1 + \alpha_0/\big(2\sqrt{\alpha_0-1}\,\big)\ln t_1\Big]/\Big(\sqrt{1+\alpha_0^2}\ln t_1\Big) = 0.$$

We have thereby derived the first determining property $\underline{\ln}_x(p', d) = 0$ of a lower characteristic degree for the vector $d \in D$. The second determining property (2_2) is also realized in the direction $t_2 = t_1^{\alpha_0}$ for sufficiently large t_1. We have thereby proved the inclusion $D \subset \underline{D}_x(p')$.

Let us prove the opposite inclusion $\underline{D}_x(p') \subset D$. We choose an arbitrary vector $d \in \underline{D}_x(p')$. Then $\lim_{t_1=1, t_2\to\infty}\left(\ln\left(1+e^{1-t_2}\right) - d_2\ln t_2\right)/\ln t_2 = -d_2 \geq 0$, since otherwise we would arrive at a contradiction with (2_1).

Let us prove the strict inequality $d_2 > 0$. Suppose the contrary: $d_2 = 0$. Then relation (2_1) acquires the form

$$\underline{\lim}_{t\to\infty}\frac{\ln\left(1+e^{t_1-t_2}\right) + \ln\psi(t) - d_1\ln t_1}{\|\ln t\|} = 0. \tag{8}$$

If d_1 is strictly less than zero, then in the direction $t_2 = t_1^{d_1^2+1}$, where t_1 is large enough to ensure that the function $\ln\psi(t)$ coincides with the function $v(t)$ in this direction, we have the estimates

$$\lim_{t_2=t_1^{d_1^2+1}\to\infty}\frac{\ln\left(1+e^{t_1-t_1^{d_1^2+1}}\right) - d_1\ln t_1 - d_1\ln t_1}{\sqrt{1+\left(d_1^2+1\right)^2}\ln t_1} = \frac{-2d_1}{\sqrt{1+\left(d_1^2+1\right)^2}} < 0,$$

which contradict (8). If the first component d_1 of the vector d is strictly less than zero, then, in a similar way, in the direction $t_2 = t_1^{4d_1^2+1}$, where t_1 is sufficiently large, we obtain the inequalities

$$\lim_{t_2=t_1^{4d_1^2+1}\to\infty}\frac{\ln\left(1+e^{t_1-t_1^{4d_1^2+1}}\right) - \sqrt{4d_1^2}\ln t_1 - d_1\ln t_1}{\sqrt{1+\left(4d_1^2+1\right)^2}\ln t_1} = \frac{d_1}{\sqrt{1+\left(4d_1^2+1\right)^2}} < 0,$$

which again contradict (8). Considering the direction $t_2 = t_1^2$ with a sufficiently large t_1, we find that d_1 also cannot vanish. Therefore, $d_2 < 0$.

Then, choosing the direction $t_2 = t_1^{\alpha_0}$, $\alpha_0 = 1 + 1/\left(4d_2^2\right) > 1$ [with t_1 large enough to ensure that $\ln\psi(t) = v(t)$ in this direction], we obtain the estimates

$$\lim_{t_2=t_1^{\alpha_0}\to\infty}\frac{\ln\left(1+e^{t_1-t_1^{\alpha_0}}\right) - \sqrt{1/\left(4d_2^2\right)}\ln t_1 - d_1\ln t_1 - d_2\left(1+1/\left(4d_2^2\right)\right)\ln t_1}{\sqrt{1+\alpha_0^2}\ln t_1}$$
$$= \frac{1/\left(2d_2\right) - d_1 - d_2 - 1/\left(4d_2\right)}{\sqrt{1+\alpha_0^2}} \geq 0,$$

since otherwise we would arrive at a contradiction with condition (2_1). It follows from these estimates that $-d_1 \geq d_2\left(1 - 1/\left(4d_2^2\right)\right)$. If this inequality were nonstrict, then there would exist an $h > 0$ such that $-d_1 = d_2\left(1 - 1/\left(4d_2^2\right)\right) + h$. Then for $\varepsilon_0 = h/2 > 0$, we would have the inequalities

$$\underline{\lim}_{t\to\infty}\frac{\ln\left(1+e^{t_1-t_2}\right) + \ln\psi(t) + d_2\left(1-1/\left(4d_2^2\right)\right)\ln t_1 - d_2\ln t_2 + (h/2)\ln t_1}{\|\ln t\|}$$
$$\geq \underline{\lim}_{t\to\infty}\frac{\ln\left(1+e^{t_1-t_2}\right) + \ln\psi(t) + d_2\left(1-1/\left(4d_2^2\right)\right)\ln t_1 - d_2\ln t_2}{\|\ln t\|} + \underline{\lim}_{t\to\infty}\frac{(h/2)\ln t_1}{\|\ln t\|} \geq 0,$$

which contradict (2_2), since the parametrization $d_1 = (2-\alpha_0)/\left(2\sqrt{\alpha_0 - 1}\,\right)$, $d_2 = -1/\left(2\sqrt{\alpha_0-1}\,\right)$ completely fills the curve $-d_1 = d_2\,(1 - 1/(4d_2^2))$, $d_2 < 0$.

Therefore, any lower characteristic degree $d = (d_1, d_2) \in \underline{D}_x\,(p')$ lies on the curve $-d_1 = d_2\,(1 - 1/(4d_2^2))$, $d_2 < 0$, and hence belongs to the set D.

Obviously, the vector $d = (d_1, d_2) \in \underline{D}_x\,(p')$ satisfies the inequalities $-\infty < d_1 < +\infty$ and $-\infty < d_2 < 0$; therefore, the above-constructed Pfaff equation satisfies the assumptions of Theorem 1. The proof of the theorem is complete.

The following assertion gives necessary properties of the boundary lower degree set.

Theorem 2. *Suppose that the lower characteristic set P_x of a solution $x(t) \neq 0$ of system* (1) *consists of more than one point. Then the nonempty left* (*respectively, right*) *boundary lower degree set $\underline{D}_x\,(p')$* [*respectively, $\underline{D}_x\,(p'')$*] *of this solution is a closed concave monotone decreasing right- and lower-unbounded* (*respectively, left- and upper-unbounded*) *curve on the two-dimensional plane and has a negative slope ≥ -1* (*respectively, ≤ -1*).

Proof. The proof of this theorem is based on Lemmas 2–4 below.

Lemma 2. *A nonempty boundary lower degree set of any nontrivial solution $x(t)$ of a completely integrable Pfaff system* (1) *with bounded continuously differentiable coefficients is a closed concave monotone decreasing curve on the two-dimensional plane.*

This lemma can be proved by analogy with [3, 8].

Lemma 3. *A nonempty left* (*respectively, right*) *boundary lower degree set $\underline{D}_x\,(p')$* [*respectively, $\underline{D}_x\,(p'')$*] *of a nontrivial solution $x(t)$ of the Pfaff system* (1) *whose lower characteristic set P_x consists of more than one point is an curve lying in the two-dimensional plane od_1d_2, right* (*respectively, left*) *unbounded with respect to d_1, and lower* (*respectively, upper*) *unbounded with respect to d_2.*

Proof. Let us prove the assertion of Lemma 3 for a left boundary lower degree set. The proof for a right boundary lower degree set is similar.

Since the left boundary lower degree set $\underline{D}_x\,(p')$ is nonempty, it follows that there exists a point $d^0 = (d_1^0, d_1^0) \in \underline{D}_x\,(p')$.

Once we prove the existence of a number $d_2 \in R$ such that $(d_1, d_2) \in \underline{D}_x\,(p')$ for every $d_1 \in R$, $d_1^0 < d_1$, it will follow that the curve $\underline{D}_x\,(p')$ is right unbounded with respect to d_1. The lower unboundedness of the curve $\underline{D}_x\,(p')$ with respect to d_2 follows from the consideration of all possible forms of closed concave monotone decreasing curves with right-unbounded first component.

Since the boundary lower degree set is nonempty, we have a function $\underline{\ln}_x\,(p', \cdot) : R^2 \to R$, $\underline{\ln}_x\,(p', d) \in R$. It follows from the definition of this function that it is continuous with respect to $d = (d_x, d_2)$ and, moreover, satisfies the Lipschitz condition with constant 1.

We choose an arbitrary number $d_1 \in R$ such that $d_1^0 < d_1$. Then there exists an $\varepsilon > 0$ such that $d_1 = d_1^0 + \varepsilon$.

Let us show that the function $\underline{\ln}_x\,(p', (d_1^0 + \varepsilon, d_2))$, continuous with respect to d_2 on the closed interval $[d_2^0 - \gamma, d_2^0]$, $\gamma > \varepsilon$, takes values of opposite signs at the endpoints of this interval. Since the lower characteristic set P_x consists of more than one point and p' is its left boundary point, it follows from Lemma 1 that there exists a sequence $\{t(k)\} \uparrow +\infty$ realizing the lower limit $\underline{\ln}_x\,(p', (d_1^0 + \varepsilon, d_2^0 - \gamma))$ and satisfying the inequalities $0 \leq \lim_{k\to\infty} \ln t_1(k)/\left\|\ln t_2(k)\right\| \leq 1$.

By virtue of (2_1), without loss of generality, we can assume that either

$$\lim_{k\to\infty} R_x^1\left(p', d^0, t(k)\right) = +\infty \qquad \text{or} \qquad \lim_{k\to\infty} R_x^1\left(p', d^0, t(k)\right) \geq 0.$$

If we suppose that the first case takes place, then we obtain $\underline{\ln}_x(p', (d_1^0 + \varepsilon, d_2^0 - \gamma)) = +\infty$. Hence the lower limit $\underline{\ln}_x\,(p', d^0)$ is equal to $+\infty$, which contradicts (2_1). Therefore, for the value of the function $\underline{\ln}_x(p', (d_1^0 + \varepsilon, d_2))$ at the left endpoint of the interval $[d_2^0 - \gamma, d_2^0]$, we have the lower bound

$$\underline{\ln}_x\left(p', \left(d_1^0 + \varepsilon, d_2^0 - \gamma\right)\right)$$
$$= \varliminf_{k\to\infty} R_x^1\left(p', d^0, t(k)\right) + \left(-\varepsilon \lim_{k\to\infty} \ln t_1(k)/\ln t_2(k) + \gamma\right) \Big/ \sqrt{\left(\lim_{k\to\infty} \ln t_1(k)/\ln t_2(k)\right)^2 + 1} > 0.$$

From inequality (2_1), we find that the value of the function $\underline{\ln}_x(p', (d_1^0 + \varepsilon, d_2^0))$ at the right endpoint of the closed interval $[d_2^0 - \gamma, d_2^0]$ admits the estimate $\underline{\ln}_x(p', (d_1^0 + \varepsilon, d_2^0)) < 0$. By the theorem on intermediate values, there exists a point $d_2^0 - \gamma_0 \in [d_2^0 - \gamma, d_2^0]$ at which the function $\underline{\ln}_x(p', (d_1^0 + \varepsilon, d_2))$ vanishes. Consequently, the point $(d_1^0 + \varepsilon, d_2^0 - \gamma_0)$ satisfies the first determining property $\underline{\ln}_x(p', (d_1^0 + \varepsilon, d_2^0 - \gamma_0)) = 0$ of the characteristic degree.

Let us prove the second determining property of the lower characteristic degree for this point. Let the lower limit $\underline{\ln}_x(p', (d_1^0 + \varepsilon, d_2^0 - \gamma_0))$ be realized on the sequence $\{\tau(k)\} \uparrow +\infty$ defined in Lemma 1. If $\ln\tau_1(k)/\ln\tau_2(k) \to a$, $0 < a \in R$, then the inequalities $\underline{\ln}_x(p', (d_1^0 + \varepsilon + \tilde{\varepsilon}, d_2^0 - \gamma_0)) < 0$ and $\underline{\ln}_x(p', (d_1^0 + \varepsilon, d_2^0 - \gamma_0 + \tilde{\varepsilon})) < 0$, $\tilde{\varepsilon} > 0$, are valid on this sequence. We have thereby shown that the point $(d_1^0 + \varepsilon, d_2^0 - \gamma_0) \in \underline{D}_x(p')$ belongs to the left boundary lower degree set.

If we suppose that $\ln\tau_1(k)/\ln\tau_2(k) \to 0$ as $k \to \infty$, then, without loss of generality, we obtain the inequalities

$$0 = \underline{\ln}_x\left(p', \left(d_1^0 + \varepsilon, d_2^0 - \gamma_0\right)\right) \le \underline{\ln}_x\left(p', \left(d_1^0, d_2^0 - \gamma_0\right)\right) \le \varliminf_{k\to\infty} R_x^1\left(p', \left(d_1^0, d_2^0 - \gamma_0\right), \tau(k)\right)$$
$$= \varliminf_{k\to\infty} R_x^1\left(p', \left(d_1^0 + \varepsilon, d_2^0 - \gamma_0\right), \tau(k)\right) + \varepsilon \lim_{k\to\infty} \frac{\ln\tau_1(k)}{\|\ln\tau(k)\|} = 0.$$

It follows from these inequalities that $\underline{\ln}_x(p', (d_1^0, d_2^0 - \gamma_0))$ is equal to 0 and the lower limit $\underline{\ln}_x(p', (d_1^0, d_2^0 - \gamma_0))$ is realized by the sequence $\{\tau(k)\}$. Therefore, without loss of generality, we obtain the chain of contradictory inequalities

$$0 = \underline{\ln}_x\left(p', \left(d_1^0, d_2^0\right)\right) \le \varliminf_{k\to\infty} R_x^1\left(p', \left(d_1^0, d_2^1 - \gamma_0\right), \tau(k)\right) - \gamma_0 \lim_{k\to\infty} \frac{\ln\tau_2(k)}{\|\ln\tau(k)\|} = -\gamma_0 < 0.$$

Consequently, the sequence $\{\tau(k)\}$ satisfies the condition $\lim_{k\to\infty} \ln\tau_1(k)/\ln\tau_2(k) = a > 0$, $a \in R$.

Therefore, for each number $d_1 = d_1^0 + \varepsilon > d_1^0$, there exists a number $d_2 = d_2^0 - \gamma_0$, $\gamma_0 > \varepsilon$, such that the vector $(d_1, d_2) \in \underline{D}_x(p')$ belongs to the left boundary lower degree set. This means that the left boundary lower degree set $\underline{D}_x(p')$ is right bounded with respect to d_1. The proof of Lemma 3 is complete.

Lemma 4. *Suppose that the lower characteristic set P_x of a nontrivial solution $x(t) \neq 0$ of the Pfaff system* (1) *consists of more than one point. Then the slope of an arbitrary secant of the nonempty left* (*respectively, right*) *boundary lower degree set $\underline{D}_x(p')$* [*respectively, $\underline{D}_x(p'')$*] *of this solution treated as a curve on the two-dimensional plane belongs to the interval* $[-1, 0)$ (*respectively,* $(-\infty, -1]$).

Proof. Since the left boundary lower degree set $\underline{D}_x(p')$ is nonempty, it follows from Lemma 3 that this set consists of more than one point. Let $d', d'' \in \underline{D}_x(p')$ be arbitrary points of the left boundary degree set; moreover, suppose that $d_1'' > d_1'$, $d_2' > d_2''$, and the line $d_2 = kd_1 + c$ is the secant passing through these points. Since the points d' and d'' lie on this line, we have the relations $d_2' = kd_1' + c$ and $d_2'' = kd_1'' + c$, which imply that $k = (d_2'' - d_2')/(d_1'' - d_1') < 0$.

On the other hand, since the points $d', d'' \in \underline{D}_x(p')$ belong to the left boundary lower degree set and we can choose the sequence $\{t(k)\} \uparrow +\infty$ realizing the lower limit $\underline{\ln}_x(p', d'')$ to be the same as in Lemma 1, without loss of generality, we have the chain of inequalities

$$0 = \underline{\ln}_x(p', d') \le \varliminf_{k\to\infty} \left[R_x^1(p', d'', t(k)) + (d_1'' - d_1') \frac{\ln t_1(k)}{\|\ln t(k)\|} + (d_2'' - d_2') \frac{\ln t_2(k)}{\|\ln t(k)\|}\right]$$
$$= (d_1'' - d_1') \lim_{k\to\infty} \frac{\ln t_1(k)}{\|\ln t(k)\|} + (d_2'' - d_2') \lim_{k\to\infty} \frac{\ln t_2(k)}{\|\ln t(k)\|} \le \frac{(d_1'' - d_1') + (d_2'' - d_2')}{\sqrt{2}}.$$

These inequalities imply the desired estimate $k = (d_2'' - d_2') / (d_1'' - d_1') \geq -1$ for the slope of the secant of the curve $\underline{D}_x(p')$.

In a similar way [with the only difference that one must consider a sequence $\{\tau(k)\} \uparrow +\infty$ realizing the lower limit $\underline{\ln}_x(p'', d')$], we can show that the slope of the secant of the curve $\underline{D}_x(p'')$ belongs to the interval $(-\infty, -1]$. The proof of the lemma is complete.

Corollary. The slope of an arbitrary tangent of the left (respectively, right) boundary lower degree set treated as a curve on the two-dimensional plane is negative and is not less than -1 (respectively, not greater than -1).

The proof of Theorem 2 is complete.

The following assertion refines necessary properties of the boundary lower degree set.

Theorem 2.1. *Suppose that the lower characteristic set* P_x *of a solution* $x(t) \neq 0$ *of system* (1) *consists of more than one point. Then the nonempty left* (*respectively, right*) *boundary lower degree set* $\underline{D}_x(p')$ [*respectively,* $\underline{D}_x(p'')$] *of this solution is a closed concave monotone decreasing right and lower unbounded* (*respectively, left and upper unbounded*) *curve on the two-dimensional plane with negative slope* ≥ -1 (*respectively,* ≤ -1) *of an arbitrary tangent and has one of the following three forms*:

(1) *unbounded on the left* (*respectively, below*) *and bounded above* (*respectively, on the right*);

(2) *unbounded on the left and above* (*respectively, on the right and below*);

(3) *bounded on the left and above* (*respectively, on the right and below*).

Moreover, all three possibilities are realized for the left (respectively, right) boundary lower degree set $\underline{D}_x(p')$ [respectively, $\underline{D}_x(p'')$] [see Theorem 1 and formulas (3) and (4)]. In this connection, the following assertion, establishing a criterion for the left (respectively, right) boundary lower degree set to be unbounded above (respectively, on the right), is of interest.

Theorem 3. *Let the set* P_x *consist of more than one point. The nonempty left* (*respectively, right*) *boundary lower degree set* $\underline{D}_x(p')$ [*respectively,* $\underline{D}_x(p'')$] *treated as a curve on the two-dimensional plane is unbounded above* (*respectively, on the right*) *if and only if there exists an infinite limit* $\lim_{\substack{t\to\infty \\ \ln t_2/\ln t_1 \to \infty}} R_x^1(p', 0, t) = +\infty$ [*respectively,* $\lim_{\substack{t\to\infty \\ \ln t_1/\ln t_2 \to \infty}} R_x^1(p'', 0, t) = +\infty$].

Proof. Let us prove Theorem 3 for the left boundary lower degree set. The right boundary lower degree set can be considered in a similar way.

Let us first prove the necessity of the assumptions of Theorem 3. We argue by contradiction. Suppose that the left boundary lower degree set $\underline{D}_x(p')$ is unbounded above and the limit fails to exist, i.e., $\lim_{\substack{t\to\infty \\ \ln t_2/\ln t_1 \to \infty}} R_x^1(p', 0, t) = +\infty$. Then there exists a sequence $\{t(k)\} \uparrow +\infty$ satisfying the conditions $\lim_{k\to\infty} \ln t_2(k)/\ln t_1(k) \equiv \alpha(t(k)) = +\infty$ and $\lim_{k\to\infty} R_x^1(p', 0, t(k)) \in R$. We choose an arbitrary point $d = (d_1, d_2) \in \underline{D}_x(p')$ of the left boundary lower degree set $\underline{D}_x(p')$. By the definition of the lower characteristic degree, for this point, we obtain the relation $\underline{\ln}_x(p', d) = 0$. Therefore, the sequence $\{t(k)\}$ must satisfy the inequality $\lim_{k\to\infty} R_x^1(p', d, t(k)) = \lim_{k\to\infty} R_x^1(p', 0, t(k)) - d_2 \geq 0$, which is equivalent to the upper boundedness of the left boundary lower degree set with respect to d_2. We have thereby arrived at a contradiction.

Let us now prove the sufficiency of the assumptions of Theorem 3. By the definition of a limit, for an arbitrary number $l_0 > 0$, there exists a number $\beta_0 > 0$ such that

$$R_x^1(p', 0, t) \geq l_0 \tag{9}$$

for all $t \in R^2_{\geq 1}$ satisfying the conditions $\|t\| \geq \beta_0$ and $\alpha(t) \geq \beta_0$.

Since the left boundary lower degree set $\underline{D}_x(p')$ is nonempty, it follows that there exists a degree $d^0 = (d_1^0, d_2^0) \in \underline{D}_x(p')$. The upper unboundedness of the set $\underline{D}_x(p')$ with respect to d_2 means that, for every $\varepsilon_0 > 0$, there exists a $d_1 \in R$ such that $(d_1, d_2^0 + \varepsilon_0) \in \underline{D}_x(p')$.

By the definition of a lower characteristic degree $d^0 \in \underline{D}_x(p')$, we obtain the relation $\underline{\ln}_x(p', d^0) = 0$ and the inequality

$$\underline{\ln}_x\left(p', (d_1^0, d_2^0 + \varepsilon_0)\right) < 0. \tag{10}$$

Let us show that there exists a number $-\delta < \min\{0, d_1^0\}$ such that

$$\underline{\ln}_x \left(p', \left(-\delta, d_2^0 + \varepsilon_0\right)\right) > 0. \tag{11}$$

Without loss of generality, let the limit $\underline{\ln}_x (p', (-\delta, d_2^0 + \varepsilon_0))$ be realized by a sequence $\{t(k,\delta)\}$ such that $\alpha(t(k,\delta)) \to \alpha(\delta)$ as $k \to \infty$. Obviously, $\alpha(\delta)$ is finite for every δ. If we suppose the contrary, namely, $\alpha(\delta) = +\infty$, then we obtain the relation $\lim_{k\to\infty} R_x^1 (p', 0, t(k,\delta)) = +\infty$, whence it follows that the limit $\underline{\ln}_x (p', (-\delta, d_2^0 + \varepsilon_0))$ is $+\infty$ and so $\underline{\ln}_x (p', d^0) = +\infty$, which contradicts the inclusion $d^0 \in \underline{D}_x (p')$.

Without loss of generality, we have the estimates

$$\begin{aligned}
&\underline{\ln}_x \left(p', \left(-\delta, d_2^0 + \varepsilon_0\right)\right) = \lim_{k\to\infty} R_x^1 \left(p', \left(-\delta, d_2^0 + \varepsilon_0\right), t(k,\delta)\right) \\
&= \lim_{k\to\infty} R_x^1 (p', 0, t(k,\delta)) + \delta \lim_{k\to\infty} \frac{\ln t_1(k,\delta)}{\|\ln t(k,\delta)\|} - \left(d_2^0 + \varepsilon_0\right) \lim_{k\to\infty} \frac{\ln t_2(k,\delta)}{\|\ln t(k,\delta)\|} \\
&\geq \lim_{k\to\infty} R_x^1 (p', 0, t(k,\delta)) + \delta/\sqrt{1+\alpha^2(\delta)} - \left|d_2^0 + \varepsilon_0\right|.
\end{aligned}$$

Without loss of generality, we assume that either $\delta/\sqrt{1+\alpha^2(\delta)} \to +\infty$ as $\delta \to +\infty$, or $\delta/\sqrt{1+\alpha^2(\delta)} \to \gamma \geq 0$ as $\delta \to +\infty$. But if there is no limit of the function $\delta/\sqrt{1+\alpha^2(\delta)}$ as $\delta \to +\infty$, then, instead of δ, we choose a sequence $\{\delta_n\}$, $\delta_n \to +\infty$ as $n \to \infty$, and perform all the forthcoming considerations for this sequence.

In the first case, there exists a number δ_0 such that $\delta/\sqrt{1+\alpha^2(\delta)} \geq |\underline{\ln}_x (p', 0)| + |d_2^0 + \varepsilon_0|$ for all $\delta \geq \delta_0$. Therefore, we have the estimate

$$\underline{\ln}_x \left(p', \left(-\delta, d_2^0 + \varepsilon_0\right)\right) \geq -|\underline{\ln}_x (p', 0)| + \delta/\sqrt{1+\alpha^2(\delta)} - \left|d_2^0 + \varepsilon_0\right| > 0 \qquad \forall \delta \geq \delta_0.$$

In the second case, we have $\alpha(\delta) \to +\infty$ as $\delta \to +\infty$. We choose an arbitrary number l_0 such that $l_0 > |d_2^0 + \varepsilon_0|$. For this number l_0, we choose a $\beta_0 > 0$ from (9), and for this β_0, there exists a number δ_0 such that $\alpha(\delta) > \beta_0$ for all $\delta \geq \delta_0$. Since $\alpha(t(k,\delta_0)) \to \alpha(\delta_0)$ as $k \to +\infty$, it follows that there exists a $k_0 = k(\delta_0)$ such that $\alpha(t(k,\delta_0)) \geq \beta_0$ for all $k \geq k_0$, and since $\|t(k,\delta_0)\| \to +\infty$ as $k \to \infty$, we have $\|t(k,\delta_0)\| \geq \beta_0$ for all $k \geq \tilde{k}(\delta_0)$. We choose a number $k' = \max\{k(\delta_0), \tilde{k}(\delta_0)\}$, and from (9), we obtain the lower estimate $R_x^1 (p', 0, t(k,\delta_0)) \geq l_0$ for all $k \geq k'$. Then $\underline{\ln}_x (p', (-\delta_0, d_2^0 + \varepsilon_0)) \geq l_0 + \delta/\sqrt{1+\alpha^2(\delta)} - |d_2^0 + \varepsilon_0| > 0$. Since the function $\underline{\ln}_x (p', (d_1, d_2^0 + \varepsilon_0))$ is continuous with respect to d_1 on the closed interval $[-\delta_0, d_1^0]$, it follows from (10) and (11) and the theorem on intermediate values that there exists a number $\tilde{d}_1 \in [-\delta_0, d_1^0]$ such that

$$\underline{\ln}_x \left(p', \left(\tilde{d}_1, d_2^0 + \varepsilon_0\right)\right) = 0. \tag{12}$$

Let us show that

$$\underline{\ln}_x \left(p', \left(\tilde{d}_1 + \tilde{\varepsilon}, d_2^0 + \varepsilon_0\right)\right) < 0 \qquad \forall \tilde{\varepsilon} > 0. \tag{13}$$

Let the lower limit $\underline{\ln}_x \left(p', \left(\tilde{d}_1, d_2^0 + \varepsilon_0\right)\right)$ be realized by a sequence $\{\tau(k)\} \uparrow +\infty$; by Lemma 1, we can suppose that this sequence satisfies the inequalities

$$0 \leq \lim_{k\to\infty} \ln \tau_1(k)/\ln \tau_2(k) \leq 1.$$

If $\lim_{k\to\infty} \ln \tau_1(k)/\ln \tau_2(k) > 0$, then we obtain the estimate

$$\underline{\ln}_x \left(p', \left(\tilde{d}_1 + \tilde{\varepsilon}, d_2^0 + \varepsilon_0\right)\right) \leq \lim_{k\to\infty} R_x^1 \left(p', \left(\tilde{d}_1 + \tilde{\varepsilon}, d_2^0 + \varepsilon_0\right), \tau(k)\right) < 0,$$

which implies (13). But if $\lim_{k\to\infty} \ln \tau_1(k)/\ln \tau_2(k) = 0$, then $\lim_{k\to\infty} R_x^1 (p', 0, \tau(k)) = +\infty$; therefore, $\underline{\ln}_x \left(p', \left(\tilde{d}_1, d_2^0 + \varepsilon_0\right)\right) = +\infty$, which contradicts (12).

We have thereby proved the inclusion $\left(\tilde{d}_1, d_2^0+\varepsilon_0\right) \in \underline{D}_x(p')$, which means that the left boundary lower degree set is unbounded above. The proof of Theorem 3 is complete.

Remark. If the lower degree set $\underline{D}_x(p)$ of a nontrivial solution $x(t)$ of system (1) is nonempty, then for the lower characteristic vector $p \in P_x$, there exists a finite limit $\underline{\ln}_x(p,0) = \underline{c}_x(p)/\sqrt{2}$.

We have shown that a nonempty boundary lower degree set of a nontrivial solution of the Pfaff system (1) can either coincide with some line $d_1 + d_2 = \underline{c}_x(p)$ of the two-dimensional plane or contain no segment of this line. Therefore, it is of interest to consider a criterion for a segment of the line $d_1 + d_2 = \underline{c}_x(p)$ to belong to a boundary lower degree set; such a criterion is given in the following assertion.

Theorem 4. *Let the lower characteristic set* P_x *of a nontrivial solution* $x(t)$ *of system* (1) *consist of more than one point. The half-line* $D \equiv \{d \in R^2 :\ d_1 + d_2 = \underline{c}_x(p'),\ d_1 \geq \max\{\underline{c}_x(p'), 0\}\}$ [*respectively, the half-line* $D \equiv \{d \in R^2 :\ d_1 + d_2 = \underline{c}_x(p''),\ d_1 \leq \min\{\underline{c}_x(p''), 0\}\}$] *of the line* $d_1 + d_2 = \underline{c}_x(p')$ [*respectively, of the line* $d_1 + d_2 = \underline{c}_x(p'')$] *belongs to the left* (*respectively, right*) *boundary lower degree set* $\underline{D}_x(p')$ [*respectively,* $\underline{D}_x(p'')$] *of this solution if and only if there exists a sequence* $\{t(k)\} \uparrow +\infty$ *realizing the lower limit* $\underline{\ln}_x(p',0)$ [*respectively,* $\underline{\ln}_x(p'',0)$] *and satisfying the condition* $\lim_{k\to\infty} \ln t_1(k)/\|\ln t(k)\| = 1/\sqrt{2}$.

Proof. Let us first prove the sufficiency of the assumptions of the theorem. To this end, we choose an arbitrary point $d \in D$ of the half-line D and show that it lies in the left boundary lower degree set $\underline{D}_x(p')$. Let the lower limit $\underline{\ln}_x(p',d)$ be realized by the sequence $\{\tau(k)\} \uparrow +\infty$ defined in Lemma 1. Then

$$0 \leq \lim_{k\to\infty} \ln\tau_1(k)/\|\ln\tau(k)\| \leq 1/\sqrt{2}, \qquad 1/\sqrt{2} \leq \lim_{k\to\infty} \ln\tau_2(k)/\|\ln\tau(k)\| \leq 1.$$

By virtue of the remark, without loss of generality, we can assume that either there exists a finite limit $\lim_{k\to\infty} R_x^1(p',0,\tau(k)) \geq \underline{c}_x(p')/\sqrt{2}$ or $\lim_{k\to\infty} R_x^1(p',0,\tau(k)) = +\infty$. We obtain the inequalities

$$\begin{aligned}\underline{\ln}_x(p',d) &= \lim_{k\to\infty} R_x^1(p',0,\tau(k)) - d_1 \lim_{k\to\infty}\frac{\ln\tau_1(k)}{\|\ln\tau(k)\|} + (d_1 - \underline{c}_x(p'))\lim_{k\to\infty}\frac{\ln\tau_2(k)}{\|\ln\tau(k)\|}\\ &\geq \frac{\underline{c}_x(p')}{\sqrt{2}} - \frac{d_1}{\sqrt{2}} + \frac{d_1 - \underline{c}_x(p')}{\sqrt{2}} = 0, \qquad d \in D,\end{aligned}$$

in the first case and the relation $\underline{\ln}_x(p',d) = +\infty$ in the second case. On the sequence $\{t(k)\} \uparrow +\infty$, we have $\lim_{k\to\infty} R_x^1(p',d,t(k)) = \underline{c}_x(p')/\sqrt{2} - d_1/\sqrt{2} + (d_1 - \underline{c}_x(p'))/\sqrt{2} = 0$, which completes the proof of the first determining property (2_1) of the lower characteristic degree for a point $d \in D$ of the half-line D. The second determining property (2_2) of a lower characteristic degree can again be proved with the use of the sequence $\{t(k)\}$ again. This completes the proof of the fact that the half-line D belongs to the left boundary lower degree set $\underline{D}_x(p')$.

Let us now prove the necessity of the assumptions of the theorem. Let the half-line D lie in the left boundary lower degree set $\underline{D}_x(p')$. Then relation (2_1) is valid for every point $d \in D$ of this half-line.

Let us show that the lower limit $\underline{\ln}_x(p',0)$ is realized on every sequence $\{t(k)\} \uparrow +\infty$ realizing the lower limit $\underline{\ln}_x(p',d) = 0$, $d \in D$, and satisfying the assumptions of Lemma 1. Suppose the contrary; namely, let there exist a sequence $\{t(k)\} \uparrow +\infty$ that realizes the lower limit $\underline{\ln}_x(p',d) = 0$, $d \in D$, satisfies assumptions of Lemma 1, and does not realize the lower limit $\underline{\ln}_x(p',0)$. Since the lower limit $\underline{\ln}_x(p',0)$ is not realized on the sequence $\{t(k)\}$, without loss of generality, we can assume that either there exists a finite limit $\lim_{k\to\infty} R_x^1(p',0,t(k)) > \underline{c}_x(p')/\sqrt{2}$ or $\lim_{k\to\infty} R_x^1(p',0,t(k)) = +\infty$. We obtain the inequalities

$$\begin{aligned}0 = \underline{\ln}_x(p',d) &= \lim_{k\to\infty} R_x^1(p',0,t(k)) - d_1 \lim_{k\to\infty}\frac{\ln t_1(k)}{\|\ln t(k)\|} + (d_1 - \underline{c}_x(p'))\lim_{k\to\infty}\frac{\ln t_2(k)}{\|\ln t(k)\|}\\ &> \frac{\underline{c}_x(p')}{\sqrt{2}} - \frac{d_1}{\sqrt{2}} + \frac{d_1 - \underline{c}_x(p')}{\sqrt{2}} = 0\end{aligned}$$

in the first case and the relation $\underline{\ln}_x(p', d) = +\infty$ in the second case; the latter relation also contradicts condition (2_1). We have thereby proved that every sequence that realizes the lower limit $\underline{\ln}_x(p', d) = 0$, $d \in D$, and satisfies the assumptions of Lemma 1 realizes the lower limit $\underline{\ln}_x(p', 0)$ as well.

We choose an arbitrary point $d^0 \in D$ of the half-line D such that $d_1^0 > 0$. Let the lower limit $\underline{\ln}_x(p', d^0) = 0$ be realized by a sequence $\{t(k)\} \uparrow +\infty$ satisfying the assumptions of Lemma 1. Then the sequence $\{t(k)\}$ realizes the lower limit $\underline{\ln}_x(p', 0)$. To prove the necessity, it remains to justify the relation $\theta_1 \equiv \lim_{k\to\infty} \ln t_1(k)/\|\ln t(k)\| = 1/\sqrt{2}$. Since the lower limits $\underline{\ln}_x(p', d^0) = 0$ and $\underline{\ln}_x(p', 0) = \underline{c}_x(p')/\sqrt{2}$ are realized on the sequence $\{t(k)\}$, we have the relations

$$\begin{aligned} 0 = \underline{\ln}_x\left(p', d^0\right) &= \underline{c}_x(p')/\sqrt{2} - d_1^0 \lim_{k\to\infty} \ln t_1(k)/\|\ln t(k)\| \\ &+ \left(d_1^0 - \underline{c}_x(p')\right) \lim_{k\to\infty} \sqrt{1 - \left(\ln t_1(k)/\|\ln t(k)\|\right)^2} \\ &= \underline{c}_x(p')/\sqrt{2} - d_1^0\theta_1 + \left(d_1^0 - \underline{c}_x(p')\right)\sqrt{1-\theta_1^2} \equiv \varphi(\theta_1). \end{aligned} \tag{14}$$

Obviously, $\theta_1 = 1/\sqrt{2}$ is a root of this equation. Let us show that Eq. (14) does not have other roots. To this end, we compute the derivative of the function $\varphi(\theta_1)$:

$$\varphi'(\theta_1) = -d_1^0 + \left(d_1^0 - \underline{c}_x(p')\right)(-2\theta_1)/\sqrt{1-\theta_1^2} < 0,$$

since the point d^0 lies on the half-line D and $d_1^0 > 0$. Therefore, $\varphi(\theta_1)$ is a strictly monotone decreasing function. Consequently, Eq. (14) has the unique root $\theta_1 = 1/\sqrt{2}$, which completes the proof of Theorem 4 for the left boundary lower degree set. The right boundary lower degree set can be considered in a similar way. The proof of Theorem 4 is complete.

Theorems 2′–4′ below are the counterparts of the preceding theorems in the case of upper characteristic degrees [9].

Theorem 2′. *Let the characteristic set Λ_x of a solution $x(t) \neq 0$ of system* (1) *consist of more than one point. Then the nonempty left* (*respectively, right*) *boundary upper degree set $\bar{D}_x(\lambda')$* [*respectively, $\bar{D}_x(\lambda'')$*] *of this solution is a closed convex monotone decreasing curve on the two-dimensional plane, is unbounded on the right and below* (*respectively, on the left and above*), *and has a negative slope ≥ -1* (*respectively, ≤ -1*) *of an arbitrary tangent.*

Theorem 3′. *Let the set Λ_x consist of more than one point. The nonempty left* (*respectively, right*) *boundary upper degree set $\bar{D}_x(\lambda')$* [*respectively, $\bar{D}_x(\lambda'')$*] *treated as a curve on the two-dimensional plane is bounded on the left* (*respectively, below*) *if and only if there exists an infinite limit* $\lim_{\substack{t\to\infty \\ \ln t_1/\ln t_2\to\infty}} R_x^1(\lambda', 0, t) = -\infty$ [*respectively,* $\lim_{\substack{t\to\infty \\ \ln t_2/\ln t_1\to\infty}} R_x^1(\lambda'', 0, t) = -\infty$].

Theorem 4′. *Let the characteristic set Λ_x of a nontrivial solution $x(t)$ of system* (1) *consist of more than one point. The half-line $\{d \in R^2 : d_1 + d_2 = \bar{c}_x(\lambda'),\ d_1 \geq \max\{\bar{c}_x(\lambda'), 0\}\}$* [*respectively, the half-line $\{d \in R^2 : d_1 + d_2 = \bar{c}_x(\lambda''), d_1 \leq \min\{\bar{c}_x(\lambda''), 0\}\}$*] *of the line $d_1 + d_2 = \bar{c}_x(\lambda')$* [*respectively, of the line $d_1 + d_2 = \bar{c}_x(\lambda'')$*] *belongs to the left* (*respectively, right*) *boundary upper degree set $\bar{D}_x(\lambda')$* [*respectively, $\bar{D}_x(\lambda'')$*] *of this solution if and only if there exists a sequence $\{t(k)\} \uparrow +\infty$ realizing the upper limit $\overline{\ln}_x(\lambda', 0)$* [*respectively, $\overline{\ln}_x(\lambda'', 0)$*] *and satisfying the condition $\lim_{k\to\infty} \ln t_1(k)/\|\ln t(k)\| = 1/\sqrt{2}$.*

The assertions of these theorems are an immediate consequence of the fact that the upper characteristic degree $\bar{d}_x(\lambda)$ of a nontrivial solution $x(t)$ of system (1) corresponding to the characteristic vector $\lambda[x]$ is equal to the lower characteristic degree $\underline{d}_{1/\|x\|}(p)$ of the function $1/\|x(t)\|$ corresponding to the lower characteristic vector $p[1/\|x\|] = -\lambda[x]$ taken with the opposite sign, i.e., $\bar{d}_x(\lambda) = -\underline{d}_{1/\|x\|}(p)$, and of the validity of Theorems 2–4 for an arbitrary function $x : R^2_{\geq 1} \to R^n\backslash\{0\}$.

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