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ORDINARY
DIFFERENTIAL EQUATIONS

The Construction of a Pfaff Linear System Whose Solutions Have an Arbitrary Given Boundary Lower Exponent Set

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Consider the Pfaff linear system

$$\partial x/\partial t_i = A_i(t)x, \qquad x \in R^n, \qquad t = (t_1, t_2) \in R^2_{>1}, \qquad i = 1, 2, \tag{1}$$

with continuously differentiable bounded matrix functions $A_i(t)$ in $R^2_{>1}$ satisfying the complete integrability condition [1, pp. 14–24; 2, pp. 16–26]

$$\partial A_1(t)/\partial t_2 + A_1(t)A_2(t) = \partial A_2(t)/\partial t_1 + A_2(t)A_1(t), \qquad t \in R^2_{>1}.$$

Let P_x be the lower characteristic set [3] of a nontrivial solution $x : R^2_{>1} \to R^n \backslash \{0\}$ of system (1), and let p' be a left boundary point of P_x. The corresponding lower characteristic exponent [4] $d = d_x(p') \in R^2$ of the solution is determined by the conditions

$$\underline{\ln}_x(p', d) \equiv \varliminf_{t\to\infty} \frac{\ln\|x(t)\| - (p', t) - (d, \ln t)}{\|\ln t\|} = 0, \qquad \ln t \equiv (\ln t_1, \ln t_2) \in R^2_+, \tag{2_1}$$

$$\underline{\ln}_x(p', d + \varepsilon e_i) < 0, \qquad e_i = (2 - i, i - 1) \in R^2, \qquad \forall \varepsilon > 0, \qquad i = 1, 2. \tag{2_2}$$

Necessary properties of the left boundary lower exponent set $\underline{D}_x(p') \equiv \bigcup\{d_x(p')\}$ of a solution $x(t)$ of system (1) were obtained in [5] for the nontrivial case in which the set P_x consists of more than one point; more precisely, it was shown that a nonempty left boundary lower exponent set is a closed concave monotone decreasing right- and lower-unbounded curve on the two-dimensional plane with negative slope ≥ -1.

In the present paper, we prove the sufficiency of these properties for the complete description of a left boundary exponent set. In particular, for any curve D in the two-dimensional plane with the above-mentioned properties, we construct a Pfaff equation

$$\partial x/\partial t_1 = a(t)x, \qquad \partial x/\partial t_2 = b(t)x, \qquad x \in R, \qquad t \in R^2_{>1}, \tag{1_1}$$

with continuously differentiable bounded functions $a(t)$ and $b(t)$ satisfying the complete integrability condition

$$\partial a(t)/\partial t_2 = \partial b(t)/\partial t_1, \qquad t \in R^2_{>1}, \tag{3}$$

such that the left boundary lower exponent set $\underline{D}_x(p')$ of any nontrivial solution $x(t)$ of the equation coincides with D.

Theorem 1. *For each bounded concave monotone decreasing right- and lower-unbounded curve D on the two-dimensional plane with negative slope ≥ -1, there exists an equation (1_1) with infinitely differentiable bounded coefficients $a : R^2_{>1} \to R$ and $b : R^2_{>1} \to R$ satisfying condition (3) such that each nontrivial solution $x : R^2_{>1} \to R\backslash\{0\}$ has the left boundary lower exponent set $\underline{D}_x(p') = D$.*

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Remark. If D is bounded on the left by a finite point $\Delta(0,0)$, then the tangent to D at the endpoint $\Delta(0,0)$ is defined as the line with slope $k(0,0)$ equal to the right limit value [at the point $\Delta(0,0) \in D$] of the slopes of D at points $d \in D$.

Proof of Theorem 1. We construct the desired Pfaff equation (1_1) by constructing its nontrivial solution.

It follows from properties of the curve D that the curve can be of one of the following three forms:

(a) unbounded on the left and bounded above;

(b) unbounded on the left and above;

(c) bounded on the left and above.

1. A PARTITION OF THE CURVE D

To construct a solution $x(t)$ with left boundary lower exponent set $\underline{D}_x(p') = D$, we construct the following partition of the curve D. We take a number $\gamma > 0$. In cases (a) and (b), in which the curve D is unbounded on the left and right, the first partition D_1 consists of points $\Delta(i,1) \in D$, $i = 0,1,2$, of this curve with the first coordinates $\Delta_1(i,1) = (i-1)\gamma$, $i = 0,1,2$, respectively. The second partition $D_2 = \bigcup_{i=0}^{2\times 2^2} \{\Delta(i,2)\} \subset D$ consists of points $\Delta(i,2) \in D$ with the first components $\Delta_1(i,2) = (i-4)\gamma/2$, $i = 0,1,\ldots,2\times 2^2$. Finally, the lth partition $D_l = \bigcup_{i=0}^{l\times 2^l} \{\Delta(i,l)\} \subset D$ consists of points $\Delta(i,l) \in D$ with the first components $\Delta_1(i,l) = \left(i \times 2^{1-l} - l\right)\gamma$, $i \in \left\{0,1,\ldots,l\times 2^l\right\} \equiv I_l$. Therefore, for each subsequent partition of the domain of D, the partition interval is increased by γ on both sides, and each new partition includes all points of the previous partition as well as the midpoints of the intervals formed by neighboring points of the previous partition. By continuing the partition of the curve D unboundedly, we obtain a countable set

$$D_\infty = \bigcup_{l=1}^{+\infty} \bigcup_{i=0}^{l\times 2^l} \{\Delta(i,l)\} \subset D,$$

which is everywhere dense on D.

In case (c), in which the curve D is bounded on the left by a finite point $\Delta(0,0) \in D$, the partition D_l of this curve consists of points $\Delta(i,l) \in D$ with the first components

$$\Delta_1(i,l) = \Delta_1(0,0) + i\gamma \times 2^{1-l}, \qquad i \in I_l.$$

Finally, just as in cases (a) and (b), in a similar way, we obtain a countable set D_∞ everywhere dense on the curve $D\colon \overline{D}_\infty = D$.

We also introduce the set $D(l)$ that is the part of D lying between the points $\Delta(0,l) \in D_l$ and $\Delta\left(l\times 2^l, l\right) \in D_l$, including the points themselves.

2. THE CONSTRUCTION OF A SOLUTION

We define the desired solution $x(t)$ by the relation $x(t) = \varphi(t)\psi(t)$, where $\varphi(t) = e^{-t_1} + e^{-t_2}$. The function $\psi(t)$ is constructed in such a way that the left boundary lower exponent set of $x(t)$ coincides with D and its left characteristic set P_x satisfies the relation $P_x = P_\varphi$.

At the ith point $\Delta(i,l) \in D$, $i \in I_l$, of the lth partition, $l \in N$, we draw some tangent

$$d_2 - \Delta_2(i,l) = k(i,l)(d_1 - \Delta_1(i,l)), \qquad k(i,l) \in [-1,0), \qquad (d_1,d_2) \in R^2,$$

to D, which does not lie below the curve. The existence of such a tangent follows from the concavity of D. Moreover, if a point $\Delta \in D$ belongs to the partition, then we draw the same tangent at that point for all subsequent partitions. This will ensure the existence of a sequence providing the lower limit at the point Δ in condition (2_1) for the lower characteristic exponent.

To match different infinitely differentiable functions with the preservation of this property, we shall use the infinitely differentiable functions

$$e_{101}(\tau;\alpha_1,\alpha_2,\alpha_3)=e_{01}(\tau;\alpha_2,\alpha_3)+[1-e_{01}(\tau;\alpha_1,\alpha_2)], \tag{4}$$
$$e_{0110}(\tau;\alpha_1,\alpha_2,\alpha_3,\alpha_4)=e_{01}(\tau;\alpha_1,\alpha_2)(1-e_{01}(\tau;\alpha_3,\alpha_4)), \tag{5}$$
$$\alpha_1<\alpha_2<\alpha_3<\alpha_4,\qquad \tau\in R,$$

defined on the basis of the function [6]

$$e_{01}(\tau;\tau_1,\tau_2)=\begin{cases}\exp\left\{-(\tau-\tau_1)^{-2}\exp\left[-(\tau_2-\tau)^{-2}\right]\right\} & \text{for}\quad \tau\in(\tau_1,\tau_2)\\ [1+\operatorname{sgn}(\tau-2^{-1}(\tau_1+\tau_2))]/2 & \text{for}\quad \tau\in(-\infty,\tau_1]\cup[\tau_2,+\infty),\end{cases}$$

$-\infty<\tau_1<\tau_2<+\infty$.

For each $l\in N$ and $i\in I_l$, we introduce the function

$$\begin{aligned}\ln\psi_{i,l}(t)&\equiv(\Delta(i,l),\ln t)e_{0110}\left(\frac{\ln t_2}{\ln t_1};\Theta_{i,l}-\tau_l-\frac14,\Theta_{i,l}-\tau_l,\Theta_{i,l}+\tau_l,\Theta_{i,l}+\tau_l+\frac14\right)\\&\quad+\|\ln t\|^2e_{101}\left(\frac{\ln t_2}{\ln t_1};\Theta_{i,l}-\tau_l,\Theta_{i,l},\Theta_{i,l}+\tau_l\right),\qquad t\in R^2_{>1},\end{aligned}\tag{6}$$

$$\Theta_{i,l}\equiv1/|k(i,l)|,\qquad \tau_l\equiv\min\left\{1/2;2^{-l}\left\|\Delta\left(l\times2^l,l\right)-\Delta(0,l)\right\|^{-1}\right\},\tag{7}$$

which takes the value equal to the inner product $(\Delta(i,l),\ln t)$ in the direction $\dfrac{\ln t_2}{\ln t_1}=\Theta_{i,l}$ and is equal to $\|\ln t\|^2$ for all $t\in R^2_{>1}$ such that $\dfrac{\ln t_2}{\ln t_1}\ge\Theta_{i,l}+\tau_l+\dfrac14$ or $\dfrac{\ln t_2}{\ln t_1}\le\Theta_{i,l}-\tau_l-\dfrac14$. Note that from the definition of the function $\ln\psi_{i,l}(t)$ in neighborhoods of the coordinate axes t_1 and t_2, we eliminate the inner product $(\Delta(i,l),\ln t)$ so as to make the coefficients of Eq. (1_1) bounded.

By virtue of the definition of the set $D(l)$ and the monotone decay of the curve D, the inequalities

$$\Delta_1(0,l)\le d_1\le\Delta_1\left(l\times2^l,l\right),\qquad \Delta_2\left(l\times2^l,l\right)\le d_2\le\Delta_2(0,l)\tag{8}$$

are valid for any given $l\in N$ and for any $d\in D(l)$. By using (8), for the function $\ln\psi_{i,l}(t)$, $i\in I_l$, we obtain the estimates

$$\begin{aligned}\ln\psi_{i,l}(t)-(d,\ln t)&\ge\|\ln t\|^2-(|\Delta(i,l)|,\ln t)-\Delta_1\big(l\times2^l,l\big)\ln t_1-\Delta_2(0,l)\ln t_2\\&\ge\|\ln t\|^2-\big(c_1(l)+\Delta_1\big(l\times2^l,l\big)\big)\ln t_1-(c_2(l)+\Delta_2(0,l))\ln t_2\equiv f_l(t),\end{aligned}$$
$$c_k(l)=\max\left\{|\Delta_k(0,l)|,\left|\Delta_k\left(l\times2^l,l\right)\right|\right\},\qquad k=1,2,\qquad t\in R^2_{>1}\backslash S(i,l),$$
$$\forall d\in D(l),\qquad i\in I_l,\qquad l\in N,$$

where $S(i,l)\equiv\left\{t\in R^2_{>1}:\ \left|\dfrac{\ln t_2}{\ln t_1}-\Theta_{i,l}\right|\le\tau_l\right\}$. Since

$$\lim_{t\to\infty,\,t\in R^2_{>1}\backslash S(i,l)}f_l(t)=+\infty,$$

it follows that there exists a number $T_l\ge1$ such that

$$\ln\psi_{i,l}(t)-(d,\ln t)\ge0,\qquad t\in R^2_{>1}\backslash S(i,l),\qquad\|t\|\ge T_l,\qquad\forall d\in D(l),\qquad i\in I_l,\qquad l\in N.\tag{9}$$

By using some values $\eta_1\ge2$ and $c\ge\exp(100)$, we introduce the numbers

$$\begin{aligned}&\nu_l=c\,\Theta_l^6\big(\Delta^2(l)+\exp\left(4\tau_l^{-2}\right)\big),\qquad \varrho_l=2\left(T_l+\nu_l^{4\Theta_l}\right),\\&\Theta_l=\max_{i\in I_l}\{\Theta_{i,l}\},\qquad \Delta(l)=\max_{i\in I_l}\{\|\Delta(i,l)\|\},\end{aligned}\tag{10_1}$$

$$\alpha_{i,l}=(\eta_l+\varrho_l)\times2^{2i},\qquad\beta_{i,l}=2\alpha_{i,l},\qquad\eta_{l+1}=\beta_{l\times2^l,l}+2^{l+1},\qquad i\in I_l,\qquad l\in N,\tag{10_2}$$

as well as the "basic" strips

$$\Pi(i,l)=\left\{t\in R^2_{>1}:\ \beta_{i,l}\le t_1+t_2\le\alpha_{i+1,l}\right\},\qquad i=0,1,\ldots,l\times 2^l-1,$$
$$\Pi\left(l\times 2^l,l\right)=\left\{t\in R^2_{>1}:\ \beta_{l\times 2^l,l}\le t_1+t_2\le\alpha_{0,l+1}\right\},\qquad l\in N,$$

and the "transition" strips

$$\tilde\Pi(i,l)=\left\{t\in R^2_{>1}:\ \alpha_{i,l}<t_1+t_2<\beta_{i,l}\right\},\qquad i\in I_l,\qquad l\in N,$$

and the closed triangle $T=\{t\in R^2_{>1}:\ t_1+t_2\le\alpha_{0,1}\}$. Therefore, the whole quadrant $R^2_{>1}$ [the domain of the desired solution $x(t)$] splits into the strips $R^2_{>1}=T\cup\left(\bigcup_{l\in N}\bigcup_{i\in I_l}\left(\Pi(i,l)\cup\tilde\Pi(i,l)\right)\right)$.

By setting

$$\Pi L(i,l)\equiv\tilde\Pi(i,l)\cup\Pi(i,l)\cup\tilde\Pi(i+1,l),\qquad i=0,1,\ldots,l\times 2^l-1,\qquad l\in N,$$
$$\Pi L\left(l\times 2^l,l\right)\equiv\tilde\Pi\left(l\times 2^l,l\right)\cup\Pi\left(l\times 2^l,l\right)\cup\tilde\Pi(0,l+1),\qquad l\in N,$$
$$S\Pi(i,l)\equiv S(i,l)\cap\Pi L(i,l),\qquad i\in I_l,\qquad l\in N,$$

from (9), we obtain the estimate

$$\ln\psi_{i,l}(t)-(d,\ln t)\ge 0,\qquad \forall t\in\Pi L(i,l)\backslash S\Pi(i,l),\qquad \forall d\in D(l),\qquad i\in I_l,\qquad l\in N,\tag{11}$$

since $\|t\|\ge(t_1+t_2)/2\ge T_l$ in each strip $\Pi L(i,l)$, $i\in I_l$, $l\in N$.

Note also that in each strip $\Pi L(i,l)$, $i\in I_l$, $l\in N$, from relations (10_1) and (10_2) and from the definition of the strip, we obtain

$$\sqrt{\|t\|}\ge\sqrt{(t_1+t_2)/2}\ge\Delta(l),\qquad t\in\Pi L(i,l),\qquad i\in I_l,\qquad l\in N.\tag{12}$$

Let us now proceed to the construction of the auxiliary function $\tilde\psi(t)$. First, in the closed triangle T, we define this function as

$$\ln\tilde\psi(t)=0,\qquad t\in T.\tag{13_1}$$

In each "basic" strip $\Pi(i,l)$, $i\in I_l$, $l\in N$, we set the function $\tilde\psi(t)$ equal to the function $\psi_{i,l}(t)$, i.e.,

$$\ln\tilde\psi(t)=\ln\psi_{i,l}(t),\qquad t\in\Pi(i,l),\qquad i\in I_l,\qquad l\in N.\tag{13_2}$$

Therefore, in all "basic" strips $\Pi(i,l)$, the function $\ln\tilde\psi(t)$ is defined on the basis of the ith point $\Delta(i,l)$ of the lth partition of the curve D. In each "transition" strip $\tilde\Pi(i+1,l)$, $i=0,1,\ldots,l\times 2^l-1$, $l\in N$, we define the function $\ln\tilde\psi(t)$ by the relation

$$\begin{gathered}\ln\tilde\psi(t)=\ln\psi_{i,l}(t)+\left[\ln\psi_{i+1,l}(t)-\ln\psi_{i,l}(t)\right]e_{01}\left(\ln\sqrt{t_1+t_2};\ln\sqrt{\alpha_{i+1,l}},\ln\sqrt{\beta_{i+1,l}}\right),\\ t\in\tilde\Pi(i+1,l),\qquad i=0,1,\ldots,l\times 2^l-1,\qquad l\in N,\end{gathered}\tag{13_3}$$

and in the "global transition" strips $\tilde\Pi(0,l+1)$, $l\in N$, we set

$$\begin{gathered}\ln\tilde\psi(t)=\ln\psi_{l\times 2^l,l}(t)+\left[\ln\psi_{0,l+1}(t)-\ln\psi_{l\times 2^l,l}(t)\right]e_{01}\left(\ln\sqrt{t_1+t_2};\ln\sqrt{\alpha_{0,l+1}},\ln\sqrt{\beta_{0,l+1}}\right),\\ t\in\tilde\Pi(0,l+1),\qquad l\in N.\end{gathered}\tag{13_4}$$

Finally, in the strip $\tilde\Pi(0,1)$, we set

$$\ln\tilde\psi(t)=\ln\psi_{0,1}(t)e_{01}\left(\ln\sqrt{t_1+t_2};\ln\sqrt{\alpha_{0,1}},\ln\sqrt{\beta_{0,1}}\right),\qquad t\in\tilde\Pi(0,1).\tag{13_5}$$

In cases (a) and (b), in which the curve D is unbounded on the left and right, we define the function $\psi(t)$ by the formula $\psi(t)=\tilde{\psi}(t)$, $t\in R^2_{>1}$.

But in case (c), in which the curve D is bounded on the left by a finite point $\Delta(0,0)\in D$, to provide that the curve D belongs to the left boundary exponent set $\underline{D}_x(p')$ and that the curve $\underline{D}_x(p')$ is not larger than the given curve D, near the t_2-axis, we define the function $\ln\psi(t)$ as the inner product $(\Delta(0,0),\ln t)$; more precisely, we set

$$\ln\psi(t)=\ln\tilde{\psi}(t)+\Big[(\Delta(0,0),\ln t)-\ln\tilde{\psi}(t)\Big]\,e_{01}\left(\frac{\ln t_2}{\ln t_1};\ \frac{3}{|k(0,0)|},\ \frac{3}{|k(0,0)|}+\frac{1}{2}\right),\qquad t\in R^2_{>1}. \tag{14}$$

Since the curve D is monotone decreasing and concave, it follows that in case (c), the slopes at the point $\Delta(0,0)\in D$ and at an arbitrary point d of the curve D satisfy the inequality $0>k(\Delta(0,0))\ge k(d)$ and hence the equivalent inequality $1/|k(\Delta(0,0))|\ge 1/|k(d)|$. Therefore, for each point $d\in D_\infty$ in the direction $\dfrac{\ln t_2}{\ln t_1}=\dfrac{1}{|k(d)|}$, the function $\ln\psi(t)$ coincides with the function $\ln\tilde{\psi}(t)$, more precisely, with the inner product $(d,\ln t)$. This is necessary for the existence of a direction in which the lower limit in condition (2_1) of the lower characteristic exponent is realized.

3. THE CONSTRUCTION OF THE LOWER CHARACTERISTIC SET

Let us show that the lower characteristic set of the solution $x(t)$ of Eq. (1_1) constructed above coincides with the lower characteristic set $P_\varphi\equiv\big\{p\in R^2_-:\ p_1+p_2=-1\big\}$ of the function $\varphi(t)$. To this end, we prove the existence of the limit

$$\lim_{t\to\infty}\left(\|t\|^{-1}\ln\psi(t)\right)=0. \tag{15}$$

By (13_2), (6), and (12), the estimates

$$\begin{aligned}\Big|\ln\tilde{\psi}(t)\Big|\Big/\|t\|&=|\ln\psi_{i,l}(t)|/\|t\|\le|(\Delta(i,l),\ln t)|/\|t\|+\|\ln t\|^2/\|t\|\\&\le\Delta(l)\|\ln t\|/\|t\|+\|\ln t\|^2/\|t\|\le\|\ln t\|/\sqrt{\|t\|}+\|\ln t\|^2/\|t\|,\\&\quad t\in\Pi(i,l),\qquad i\in I_l,\qquad l\in N,\end{aligned} \tag{16_1}$$

are valid in each "basic" strip $\Pi(i,l)$, $i\in I_l$, $l\in N$. In a similar way, in each "transition" strip $\tilde{\Pi}(i+1,l)$, $i=0,1,\dots,l\times 2^l-1$, $l\in N$, from (13_3), (6), and (12), we obtain the estimates

$$\begin{aligned}\Big|\ln\tilde{\psi}(t)\Big|\Big/\|t\|&\le 2\,|\ln\psi_{i,l}(t)|/\|t\|+|\ln\psi_{i+1,l}(t)|\,/\|t\|\\&\le 3\left(\|\ln t\|/\sqrt{\|t\|}+\|\ln t\|^2/\|t\|\right),\\&\quad t\in\tilde{\Pi}(i+1,l),\qquad i=0,1,\dots l\times 2^l-1,\qquad l\in N.\end{aligned} \tag{16_2}$$

In the "transition" strips $\tilde{\Pi}(0,l+1)$, $l\in N$, from (13_4), we obtain

$$\Big|\ln\tilde{\psi}(t)\Big|\Big/\|t\|\le 3\left(\|\ln t\|/\sqrt{\|t\|}+\|\ln t\|^2/\|t\|\right),\qquad t\in\tilde{\Pi}(0,l+1),\qquad l\in N. \tag{16_3}$$

Likewise, from (13_5) in the strip $\tilde{\Pi}(0,1)$, we obtain the inequality

$$\Big|\ln\tilde{\psi}(t)\Big|\Big/\|t\|\le\|\ln t\|/\sqrt{\|t\|}+\|\ln t\|^2/\|t\|,\qquad t\in\tilde{\Pi}(0,1). \tag{16_4}$$

Therefore, it follows from (16_1)–(16_4) and (13_1) that

$$\Big|\ln\tilde{\psi}(t)\Big|\Big/\|t\|\le 3\left(\|\ln t\|/\sqrt{\|t\|}+\|\ln t\|^2/\|t\|\right),\qquad t\in R^2_{>1}, \tag{17}$$

which implies (15) in cases (a) and (b) of a curve D unbounded on the left and right.

But in case (c), in which the curve D is bounded on the left by a finite point $\Delta(0,0) \in D$, from (14) and (17), we obtain the estimates

$$|\ln\psi(t)|/\|t\| \le 2\left|\ln\tilde\psi(t)\right|\Big/\|t\| + \|\Delta(0,0)\|\|\ln t\|/\|t\|$$
$$\le 6\left(\|\ln t\|/\sqrt{\|t\|} + \|\ln t\|^2/\|t\|\right) + \|\Delta(0,0)\|\|\ln t\|/\|t\|, \qquad t \in R^2_{>1},$$

which imply that relation (15) is valid in this case as well.

We have thereby shown that the lower characteristic set P_x of the solution $x(t)$ of Eq. (1_1) coincides with the set $\left\{p \in R^2_- :\ p_1 + p_2 = -1\right\}$.

4. PROOF OF THE COINCIDENCE OF THE LEFT BOUNDARY LOWER DEGREE SET WITH A GIVEN CURVE D

We choose an arbitrary point $\tilde d = \left(\tilde d_1, \tilde d_2\right) \in D$ of the curve D. Since the partition D_∞ is dense everywhere on D, it follows that for this point, there exists a sequence $\{d(n)\}_{n\in N}$, $d(n) \in D_\infty$, of partition points converging to the point $\tilde d$, i.e., $d(n) \underset{n\to\infty}{\to} \tilde d$. Now if we prove that the partition points $d(n)$ belong to the left boundary lower exponent set $\underline{D}_x(p')$, then it will follow from its closedness [5] that the limit point $\tilde d$ also belongs to it, i.e., $\tilde d \in \underline{D}_x(p')$. This will imply the inclusion $D \subset \underline{D}_x(p')$.

Let us now show that, indeed, each point $d = (d_1, d_2) \in D_\infty$ of the partition belongs to the left boundary lower exponent set.

We introduce the notation

$$\beta(d) \equiv \varliminf_{t\to\infty} \frac{\ln|x(t)| + t_1 - (d, \ln t)}{\|\ln t\|}$$

for the lower limit in condition (2_1), where $p' = (-1, 0)$ is the left boundary point of the lower characteristic set of the constructed solution $x(t)$. Let us show that $\beta(d) = 0$.

First, we show that the limit $\beta(d)$ is nonpositive. Since the point d belongs to the countable partition D_∞ of the curve D and each new finite partition D_{l+1} contains all points of the previous finite partition D_l, it follows that there exists an index $l(d) \in N$ such that $d \in D_l$ for all $l > l(d)$ and $d \notin D_l$ for all $l \le l(d)$. If the point d coincides with the point $\Delta(0,0)$ [in case (c)], then we set $l(d) = 0$. Suppose that the point d is the i_1th point of the $(l(d)+1)$st partition, the i_2th point of the $(l(d)+2)$nd partition, and, finally, the i_mth point of the $(l(d)+m)$th partition. By definition, for each partition at the point d, we draw the same tangent with some slope $k(d)$, i.e., $\Theta_{i_1,l(d)+1} = \Theta_{i_2,l(d)+2} = \cdots = 1/|k(d)|$. In each strip $\Pi\left(i_m, l(d)+m\right)$, $m \in N$, where $\ln\psi(t)$ is a function defined on the basis of the point $d = \Delta\left(i_m, l(d)+m\right)$, we choose one point $\tau(m)$ in each closed interval $\dfrac{\ln t_2}{\ln t_1} = \dfrac{1}{|k(d)|} \ge 1$. We thereby obtain a sequence $\{\tau(m)\} \uparrow +\infty$ such that

$$\ln\psi(\tau(m)) = (d, \ln\tau(m)), \qquad \lim_{m\to\infty}\left[\ln x(\tau(m)) + \tau_1(m) - (d, \ln\tau(m))\right]/\|\ln\tau(m)\| = 0,$$

which implies that $\beta(d) \le 0$.

By $\{t(m)\} \uparrow \infty$ we denote the sequence on which the lower limit $\beta(d)$ is attained. Without loss of generality, one can assume that all elements $t(m)$ of this sequence belong to strips of the quadrant $R^2_{>1}$ with the different indices l_m, $l_m > 1$, and $l_{m+1} > l_m \to +\infty$ as $m \to +\infty$. The sequence $\{t(m)\} \uparrow \infty$ is assumed to satisfy the inclusion

$$d \in D\left(l_m\right), \qquad m \in N. \tag{18}$$

Let us prove the inequality $\beta(d) \ge 0$. If the sequence $\{t(m)\}$ contains an infinite subsequence $\{t\left(m_j\right)\}$ such that each of its points $t\left(m_j\right)$ satisfies the estimate $\ln\psi\left(t\left(m_j\right)\right) - \left(d, \ln t\left(m_j\right)\right) \ge 0$, then, obviously, the inequality $\beta(d) \geq 0$ is necessarily valid. Therefore, without loss of generality, one can assume that

$$R(m,d) \equiv \ln \psi(t(m)) - (d, \ln t(m)) < 0, \qquad \forall m \in N. \tag{19}$$

Let us consider cases (a) and (b), in which the curve D is unbounded on the right and left, and take an arbitrary $m \in N$. If $t(m)$ belongs to the "basic" strip $\Pi\,(i_m, l_m)$, then from (19) and (13_2), we obtain the inequality $\ln \psi_{i_m,l_m}(t(m)) - (d, \ln t(m)) < 0$, which, together with (11) and (18), implies that $t(m) \in S\Pi\,(i_m, l_m)$. Then from the definition of the sector $S\Pi\,(i_m, l_m)$ and from (7), we obtain the estimates

$$\left|\frac{\ln t_2(m)}{\ln t_1(m)} - \Theta_{i_m,l_m}\right| \leq \tau_{l_m} \leq 2^{-l_m}\left\|\Delta\left(l_m \times 2^{l_m}, l_m\right) - \Delta\,(0, l_m)\right\|^{-1}. \tag{20}$$

Let us now write out an equation of the tangent of the curve D at the point $\Delta\,(i_m, l_m)$:

$$\delta_2 - \Delta_2\,(i_m, l_m) = k\,(i_m, l_m)(\delta_1 - \Delta_1\,(i_m, l_m)), \qquad \delta \in R^2.$$

Since the curve D is concave, it follows that the point $d \in D$ does not lie above this tangent; therefore, $d_2 - \Delta_2\,(i_m, l_m) \leq k\,(i_m, l_m)(d_1 - \Delta_1\,(i_m, l_m))$, and consequently,

$$\Delta_1\,(i_m, l_m) - d_1 + \Theta_{i_m,l_m}\,(\Delta_2\,(i_m, l_m) - d_2) \geq 0. \tag{21}$$

Let us now estimate the quantity $R(m,d)$ below. It follows from the inclusion $t(m) \in S\Pi\,(i_m, l_m)$, (6), (21), and (20) that

$$\begin{aligned}
R(m,d) &= \ln \psi_{i_m,l_m}(t(m)) - (d, \ln t(m)) \geq (\Delta\,(i_m, l_m) - d, \ln t(m)) \\
&= (\Delta_1\,(i_m, l_m) - d_1) \ln t_1(m) + (\Delta_2\,(i_m, l_m) - d_2) \ln t_2(m) \\
&= \Bigg[\{(\Delta_1(i_m, l_m) - d_1) + \Theta_{i_m,l_m}\,(\Delta_2(i_m, l_m) - d_2)\} \\
&\quad + (\Delta_2\,(i_m, l_m) - d_2)\left(\frac{\ln t_2(m)}{\ln t_1(m)} - \Theta_{i_m,l_m}\right)\Bigg] \ln t_1(m) \\
&\geq -|\Delta_2\,(i_m, l_m) - d_2| \left|\frac{\ln t_2(m)}{\ln t_1(m)} - \Theta_{i_m,l_m}\right| \ln t_1(m) \\
&\geq -2^{-l_m}\left(|\Delta_2\,(i_m, l_m) - d_2| / \left\|\Delta\left(l_m \times 2^{l_m}, l_m\right) - \Delta\,(0, l_m)\right\|\right) \ln t_1(m) \\
&\geq -2^{-l_m}\left(|\Delta_2\,(i_m, l_m) - d_2| / \|\Delta\,(i_m, l_m) - d\|\right) \ln t_1(m) \\
&\geq -2^{-l_m} \ln t_1(m) \geq -2^{-l_m} \|\ln t(m)\|.
\end{aligned}$$

Let $t(m)$ lie in the "transition" strip $\tilde{\Pi}\,(i_m + 1, l_m)$, $i_m < l_m \times 2^{l_m}$. First, we suppose that

$$\ln \psi_{i_m+1,l_m}(t(m)) \geq \ln \psi_{i_m,l_m}(t(m))$$

at the point $t(m)$. Then it follows from (19) and (13_3) that $\ln \psi_{i_m,l_m}(t(m)) - (d, \ln t(m)) < 0$, so that, just as in the case in which $t(m)$ belongs to the "basic" strip $\Pi\,(i_m, l_m)$, one can obtain the estimate $R(m,d) \geq -2^{-l_m}\|\ln t(m)\|$. But if $\ln \psi_{i_m+1,l_m}(t(m)) < \ln \psi_{i_m,l_m}(t(m))$ at the point $t(m)$, then it follows from (19) and (13_3) that $\ln \psi_{i_m+1,l_m}(t(m)) - (d, \ln t(m)) < 0$. This, together with (11) and (18), implies that $t(m) \in S\Pi\,(i_m + 1, l_m)$. From this inclusion, from the definition of the sector $S\Pi\,(i_m + 1, l_m)$, and from (7), we obtain the estimates

$$\left|\frac{\ln t_2(m)}{\ln t_1(m)} - \Theta_{i_m+1,l_m}\right| \leq \tau_{l_m} \leq 2^{-l_m}\left\|\Delta\left(l_m \times 2^{l_m}, l_m\right) - \Delta\,(0, l_m)\right\|^{-1}. \tag{22}$$

Below we perform the same considerations as in the case of the inclusion $t(m) \in \Pi\,(i_m, l_m)$ with the only difference that we write out an equation of the tangent of the curve D not at the point $\Delta\,(i_m, l_m)$ but at the point $\Delta\,(i_m + 1, l_m)$. The desired equation has the form

$$\delta_2 - \Delta_2\,(i_m + 1, l_m) = k\,(i_m + 1, l_m)(\delta_1 - \Delta_1(i_m + 1, l_m))\,, \qquad \delta \in R^2.$$

Since the curve D is concave, it follows that the point $d \in D$ does not lie above this tangent, i.e., $d_2 - \Delta_2(i_m + 1, l_m) \le k\,(i_m + 1, l_m)(d_1 - \Delta_1(i_m + 1, l_m))$. By dividing both sides of this inequality by $(-k\,(i_m + 1, l_m))$, we obtain

$$\Delta_1\,(i_m + 1, l_m) - d_1 + \Theta_{i_m+1,l_m}\,(\Delta_2\,(i_m + 1, l_m) - d_2) \ge 0. \tag{23}$$

Let us now estimate $R(m, d)$ below with the use of (13_3), the inclusion $t(m) \in S\Pi\,(i_m + 1, l_m)$, formula (6), and inequalities (22) and (23):

$$\begin{aligned} R(m,d) &\ge \ln \psi_{i_m+1,l_m}(t(m)) - (d, \ln t(m)) \ge (\Delta(i_m + 1, l_m) - d, \ln t(m)) \\ &= \Bigg[\{(\Delta_1(i_m + 1, l_m) - d_1) + \Theta_{i_m+1,l_m}\,(\Delta_2(i_m + 1, l_m) - d_2)\} \\ &\qquad + (\Delta_2(i_m + 1, l_m) - d_2)\left(\frac{\ln t_2(m)}{\ln t_1(m)} - \Theta_{i_m+1,l_m}\right)\Bigg] \ln t_1(m) \\ &\ge -|\Delta_2\,(i_m + 1, l_m) - d_2| \left|\frac{\ln t_2(m)}{\ln t_1(m)} - \Theta_{i_m+1,l_m}\right| \ln t_1(m) \\ &\ge -2^{-l_m}\left(|\Delta_2\,(i_m + 1, l_m) - d_2| / \left\|\Delta\left(l_m \times 2^{l_m}, l_m\right) - \Delta\,(0, l_m)\right\|\right) \ln t_1(m) \\ &\ge -2^{-l_m}\|\ln t(m)\|. \end{aligned}$$

Now let $t(m)$ belong to the "global transition" strip $\tilde{\Pi}\,(0, l_m + 1)$. If

$$\ln \psi_{0,l_m+1}(t(m)) \ge \ln \psi_{l_m\times 2^{l_m},l_m}(t(m))$$

at the point $t(m)$, then it follows from (19) and (13_4) that

$$\ln \psi_{l_m\times 2^{l_m},\,l_m}(t(m)) - (d, \ln t(m)) < 0.$$

Just as in the case of the inclusion $t(m) \in \Pi\,(i_m, l_m)$, we write out the equation of the tangent of the curve D at the point $\Delta\left(l_m \times 2^{l_m}, l_m\right)$ and obtain the estimate $R(m, d) \ge -2^{-l_m}\|\ln t(m)\|$. If

$$\ln \psi_{0,l_m+1}(t(m)) < \ln \psi_{l_m\times 2^{l_m},\,l_m}(t(m)),$$

then it follows from (19) and (13_4) that $\ln \psi_{0,l_m+1}(t(m)) - (d, \ln t(m)) < 0$, which, together with the equation of the tangent at the point $\Delta\,(0, l_m + 1)$, implies the estimate

$$R(m, d) \ge -2^{-l_m-1}\|\ln t(m)\|.$$

Therefore, in cases (a) and (b), in which the curve D is unbounded on the left and right, we obtain the estimate

$$R(m, d) \ge -2^{-l_m}\|\ln t(m)\|, \qquad \forall m \in N, \qquad l_m > 1, \qquad l_m \to +\infty, \qquad m \to \infty,$$

which implies the desired property

$$\begin{aligned} \beta(d) &= \lim_{m\to\infty} \left[\ln\left(1 + e^{-t_2(m)+t_1(m)}\right) + \ln \psi(t(m)) - (d, \ln t(m))\right] / \|\ln t(m)\| \\ &\ge \lim_{m\to\infty} \left(-2^{-l_m}\right) = 0. \end{aligned}$$

The second condition (2_2) in the definition of the lower characteristic exponent for a given vector d can be proved with the use of the above-constructed sequence $\tau(m)$. We have thereby proved the inclusion $D \subset \underline{D}_x(p')$ in cases (a) and (b), and since the curve D is unbounded on the left and right, it follows that the left boundary lower exponent set $\underline{D}_x(p') = D$ coincides with D.

Let us now consider case (c), in which D is bounded on the left by a finite point $\Delta(0,0) \in D$. If either $\dfrac{\ln t_2(m)}{\ln t_1(m)} < \dfrac{3}{|k(0,0)|}$ or $\dfrac{\ln t_2(m)}{\ln t_1(m)} \geq \dfrac{3}{|k(0,0)|}$ and $\ln \tilde{\psi}(t(m)) \leq (\Delta(0,0), \ln t(m))$ for some $m \in N$, then, just as in cases (a) and (b), by virtue of the relation $\ln \psi(t(m)) \geq \ln \tilde{\psi}(t(m))$, we have the estimate $R(m,d) \geq -2^{-l_m}\|\ln t(m)\|$ for this m. But if $\dfrac{\ln t_2(m)}{\ln t_1(m)} \geq \dfrac{3}{|k(0,0)|}$, and $\ln \tilde{\psi}(t(m)) > (\Delta(0,0), \ln t(m))$, then we obtain the estimate

$$\begin{aligned} R(m,d) &\geq (\Delta(0,0) - d, \ln t(m)) = \left[(\Delta_1(0,0) - d_1) + \frac{\ln t_2(m)}{\ln t_1(m)}(\Delta_2(0,0) - d_2)\right] \ln t_1(m) \\ &\equiv (\ln t_1(m))\, f\left(\frac{\ln t_2(m)}{\ln t_1(m)}\right). \end{aligned}$$

Obviously, if the point d coincides with the point $\Delta(0,0)$, then $R(m,d) \geq 0$. Therefore, we suppose that the points d and $\Delta(0,0)$ are distinct. Since $\Delta(0,0) \in D$ is the left boundary point of the curve D and the curve D is monotone decreasing, we have $\Delta_1(0,0) - d_1 < 0$ and $\Delta_2(0,0) - d_2 > 0$. Hence we find that the function $f\left(\dfrac{\ln t_2(m)}{\ln t_1(m)}\right)$ attains its minimum value if its argument $\dfrac{\ln t_2(m)}{\ln t_1(m)}$ is minimum, i.e., if $\dfrac{\ln t_2(m)}{\ln t_1(m)} = \dfrac{3}{|k(0,0)|}$. Let us now write out the equation of the tangent $\delta_2 - \Delta_2(0,0) = k(0,0)(\delta_1 - \Delta_1(0,0))$ of the curve D at the point $\Delta(0,0)$. Since the curve D is concave, it follows that the point $d \in D$ is not above this tangent, and consequently, $(d_2 - \Delta_2(0,0)) \leq k(0,0)(d_1 - \Delta_1(0,0))$, and the equivalent inequality $\Delta_1(0,0) - d_1 + (\Delta_2(0,0) - d_2)/|k(0,0)| \geq 0$ is valid. This implies the estimates

$$R(m,d) \geq [\Delta_1(0,0) - d_1 + 3\,(\Delta_2(0,0) - d_2)/|k(0,0)|] \ln t_1(m) \geq 0.$$

We have thereby proved the estimate $R(m,d) \geq -2^{-l_m}\|\ln t(m)\|$, $m \in N$, $l_m > 1$, $l_m \to +\infty$ as $m \to \infty$, in case (c). Therefore, just as in cases (a) and (b), we have $D \subset \underline{D}_x(p')$.

Let us now show that the left boundary exponent set $\underline{D}_x(p')$ of a nontrivial solution $x(t)$ does not lie above the curve D. We choose an arbitrary point d of the left boundary exponent set $\underline{D}_x(p')$. Since the relation $\ln \psi(t') = (\Delta(0,0), \ln t')$ is valid in the direction $t' = (t'_1, t'_2)$, where $t'_1 = e$, $\ln t'_2 > 3/|k(0,0)| + 1/2$, $t'_2 \to +\infty$, and the limit

$$\lim_{t \to \infty, t = t'} \left[\ln\left(1 + e^{t'_1 - t'_2}\right) + \ln \psi(t') - (d, \ln t')\right] \Big/ \|\ln t'\| = \Delta_2(0,0) - d_2$$

is nonnegative [otherwise we would arrive at a contradiction with condition (2_1) for the lower characteristic exponent d], we have the inequality $d_2 \leq \Delta_2(0,0)$. And since the point $\Delta(0,0)$ belongs to the left boundary lower exponent set $\underline{D}_x(p')$ and the curve $\underline{D}_x(p')$ is strictly monotone decreasing, we have $\underline{D}_x(p') = D$.

5. THE CONSTRUCTION OF AN EQUATION. BOUNDEDNESS OF THE COEFFICIENTS

The constructed function $x(t) > 0$ is a solution of Eq. (1_1) with the coefficients

$$\begin{aligned} a(t) &= x^{-1}(t)\partial x(t)/\partial t_1 = \partial \ln x(t)/\partial t_1, \\ b(t) &= x^{-1}(t)\partial x(t)/\partial t_2 = \partial \ln x(t)/\partial t_2, \qquad t \in R^2_{>1}, \end{aligned}$$

satisfying the complete integrability condition (3) in view of the infinite differentiability of $\ln x(t)$ in $R^2_{>1}$.

By using the inequality

$$\left|\frac{de_{01}(\tau;\tau_1,\tau_2)}{d\tau}\right| \le 2\exp\left[2\left(\tau_2-\tau_1\right)^{-2}\right], \qquad \tau\in\left[\tau_1,\tau_2\right], \tag{24}$$

which is a simple consequence of the lemma in [7] and is valid on any closed interval $[\tau_1,\tau_2]$ of length $\tau_2-\tau_1\le 1/2$, we prove that these coefficients are bounded.

We first prove that the partial derivatives $\partial\ln\psi_{i,l}(t)/\partial t_k$, $k=1,2$, are bounded in the strip $\Pi L(i,l)$, $i\in I_l$, $l\in N$. By taking account of (6), we note that the inner product $(\Delta(i,l),\ln t)$ occurs in the definition of the function $\ln\psi_{i,l}(t)$ only in the sector

$$\tilde S(i,l)\equiv\left\{t\in R^2_{>1}:\ \Theta_{i,l}-\tau_l-\frac14\le\frac{\ln t_2}{\ln t_1}\le\Theta_{i,l}+\tau_l+\frac14\right\};$$

therefore, it suffices to prove the boundedness of the partial derivatives

$$\frac{\partial}{\partial t_k}\left[(\Delta(i,l),\ln t)e_{0110}\left(\frac{\ln t_2}{\ln t_1};\Theta_{i,l}-\tau_l-\frac14,\Theta_{i,l}-\tau_l,\Theta_{i,l}+\tau_l,\Theta_{i,l}+\tau_l+\frac14\right)\right], \qquad k=1,2,$$

in the intersection $\tilde S(i,l)\cap\Pi L(i,l)$. Since $t_1+t_2\ge 2\nu_l^{4\Theta_l}$ in the strip $\Pi L(i,l)$, $i\in I_l$, $l\in N$, and $\tau_l\le 1/2$, we have the estimates

$$t_k\ge\nu_l, \qquad 1/t_k\le 1/\nu_l, \qquad k=1,2, \tag{25}$$

in the intersection $\tilde S(i,l)\cap\Pi L(i,l)$, $i\in I_l$, $l\in N$.

Note also that, obviously, $1/\Theta_{i,l}\le 1$, $\exp\left(2\tau_l^{-2}\right)\ge 1$, and $1/\nu_l\le 1$. It follows from (25) that

$$\left|\frac{\partial}{\partial t_k}(\Delta(i,l),\ln t)\right|\le\frac{|\Delta_k(i,l)|}{t_k}\le\frac{\Delta(l)}{\nu_l}\le 1, \qquad k=1,2, \tag{26_k}$$

for $t\in\tilde S(i,l)\cap\Pi L(i,l)$, $i\in I_l$, and $l\in N$. By using (24) and (25), in the intersection

$$\tilde S(i,l)\cap\Pi L(i,l), \qquad i\in I_l, \qquad l\in N,$$

we obtain the estimates

$$\begin{aligned}&\left|(\Delta(i,l),\ln t)\frac{\partial}{\partial t_1}\left(1-e_{01}\left(\frac{\ln t_2}{\ln t_1};\Theta_{i,l}+\tau_l,\Theta_{i,l}+\tau_l+\frac14\right)\right)\right|\\ &\qquad\le\|\Delta(i,l)\|\left(\ln t_1+\ln t_2\right)\times 2e^{32}(\ln t_2)/\left(t_1\ln^2 t_1\right)\\ &\qquad\le 2e^{32}(\Delta(l)/t_1)\Big((\ln t_2)/\ln t_1+((\ln t_2)/\ln t_1)^2\Big)\le 2e^{32}(\Delta(l)/\nu_l)\left(2\Theta_l+4\Theta_l^2\right)\le 1,\end{aligned} \tag{27_1}$$

$$\begin{aligned}&\left|(\Delta(i,l),\ln t)\frac{\partial}{\partial t_2}\left(1-e_{01}\left(\frac{\ln t_2}{\ln t_1};\Theta_{i,l}+\tau_l,\Theta_{i,l}+\tau_l+\frac14\right)\right)\right|\\ &\qquad\le\|\Delta(i,l)\|(\ln t_1+\ln t_2)\times 2e^{32}/(t_2\ln t_1)\le 2e^{32}(\Delta(l)/t_2)(1+((\ln t_2)/\ln t_1))\\ &\qquad\le 2e^{32}(\Delta(l)/\nu_l)(1+2\Theta_l)\le 1,\end{aligned} \tag{27_2}$$

$$\begin{aligned}&\left|(\Delta(i,l),\ln t)\frac{\partial}{\partial t_1}e_{01}\left(\frac{\ln t_2}{\ln t_1};\Theta_{i,l}-\tau_l-\frac14,\Theta_{i,l}-\tau_l\right)\right|\\ &\qquad\le\|\Delta(i,l)\|(\ln t_1+\ln t_2)\times 2e^{32}(\ln t_2)/\left(t_1\ln^2 t_1\right)\\ &\qquad\le 2e^{32}(\Delta(l)/t_1)\Big((\ln t_2)/\ln t_1+((\ln t_2)/\ln t_1)^2\Big)\le 2e^{32}(\Delta(l)/\nu_l)\left(2\Theta_l+4\Theta_l^2\right)\le 1,\end{aligned} \tag{28_1}$$

$$\left|(\Delta(i,l),\ln t)\frac{\partial}{\partial t_2}e_{01}\left(\frac{\ln t_2}{\ln t_1};\Theta_{i,l}-\tau_l-\frac14,\Theta_{i,l}-\tau_l\right)\right|\le 2e^{32}(\Delta(l)/t_2)(1+(\ln t_2)/\ln t_1)\le 1. \tag{28_2}$$

By taking account of (6), we estimate the partial derivatives

$$\frac{\partial}{\partial t_k}\left[e_{101}\left(\frac{\ln t_2}{\ln t_1};\Theta_{i,l}-\tau_l,\Theta_{i,l},\Theta_{i,l}+\tau_l\right)\|\ln t\|^2\right],\qquad k=1,2.$$

Note that the partial derivatives $\frac{\partial}{\partial t_k}e_{101}\left(\frac{\ln t_2}{\ln t_1};\Theta_{i,l}-\tau_l,\Theta_{i,l},\Theta_{i,l}+\tau_l\right)$, $k=1,2$, vanish outside the sector $S(i,l)$. This, together with (25) and the inequalities $\frac{\ln t_2}{\ln t_1}\ge\Theta_{i,l}-\tau_l\ge\frac{1}{2}$, $t\in S(i,l)$, implies the following estimates in the strip $\Pi L(i,l)$, $i\in I_l$, $l\in N$:

$$\begin{aligned}
&\left|\left(\frac{\partial}{\partial t_1}e_{01}\left(\frac{\ln t_2}{\ln t_1};\Theta_{i,l},\Theta_{i,l}+\tau_l\right)\right)\|\ln t\|^2\right|\le 2\big(\exp(2\tau_l^{-2})\big)\frac{\ln t_2}{t_1\ln^2 t_1}\left(\ln^2 t_1+\ln^2 t_2\right)\\
&\qquad\le\big(2\big(\exp\left(2\tau_l^{-2}\right)\big)(\ln t_2)/t_1\big)\left(1+((\ln t_2)/\ln t_1)^2\right)\\
&\qquad\le\big(4\big(\exp\left(2\tau_l^{-2}\right)\big)\,\Theta_l(\ln t_1)/t_1\big)\left(1+4\Theta_l^2\right)\le\big(8\big(\exp(2\tau_l^{-2})\big)\,\Theta_l\big(\ln\sqrt{t_1}\,\big)/t_1\big)\left(1+4\Theta_l^2\right)\\
&\qquad\le\big(8\big(\exp\left(2\tau_l^{-2}\right)\big)\,\Theta_l/\sqrt{\nu_l}\,\big)\left(1+4\Theta_l^2\right)\le 1,
\end{aligned}\tag{29_1}$$

$$\begin{aligned}
&\left|\left(\frac{\partial}{\partial t_2}e_{01}\left(\frac{\ln t_2}{\ln t_1};\Theta_{i,l},\Theta_{i,l}+\tau_l\right)\right)\|\ln t\|^2\right|\le\frac{2\exp\left(2\tau_l^{-2}\right)}{t_2\ln t_1}\left(\ln^2 t_1+\ln^2 t_2\right)\\
&\qquad\le\big(2\big(\exp(2\tau_l^{-2})\big)(\ln t_1)/t_2\big)\left(1+((\ln t_2)/\ln t_1)^2\right)\le\big(4\big(\exp(2\tau_l^{-2})\big)(\ln t_2)/t_2\big)\left(1+4\Theta_l^2\right)\\
&\qquad\le\big(8\big(\exp(2\tau_l^{-2})\big)\big(\ln\sqrt{t_2}\,\big)/t_2\big)\left(1+4\Theta_l^2\right)\le\big(8\big(\exp(2\tau_l^{-2})\big)/\sqrt{\nu_l}\,\big)\left(1+4\Theta_l^2\right)\le 1,
\end{aligned}\tag{29_2}$$

$$\left|\left(\frac{\partial}{\partial t_1}\left[1-e_{01}\left(\frac{\ln t_2}{\ln t_1};\Theta_{i,l}-\tau_l,\Theta_{i,l}\right)\right]\right)\|\ln t\|^2\right|\le 2\big(\exp(2\tau_l^{-2})\big)\frac{\ln t_2}{t_1\ln^2 t_1}\left(\ln^2 t_1+\ln^2 t_2\right)\le 1,\tag{30_1}$$

$$\left|\left(\frac{\partial}{\partial t_2}\left[1-e_{01}\left(\frac{\ln t_2}{\ln t_1};\Theta_{i,l}-\tau_l,\Theta_{i,l}\right)\right]\right)\|\ln t\|^2\right|\le 2\big(\exp(2\tau_l^{-2})\big)\frac{1}{t_2\ln t_1}\left(\ln^2 t_1+\ln^2 t_2\right)\le 1.\tag{30_2}$$

Furthermore, obviously,

$$\left|e_{101}\left(\frac{\ln t_2}{\ln t_1};\Theta_{i,l}-\tau_l,\Theta_{i,l},\Theta_{i,l}+\tau_l\right)\frac{\partial}{\partial t_k}\|\ln t\|^2\right|\le 2\frac{\ln t_k}{t_k}\le 2,\qquad k=1,2.\tag{31_k}$$

It follows from definitions (6), (4), and (5) of the function $\ln\psi_{i,l}(t)$, $i\in I_l$, $l\in N$, and from the inequalities (26_k)–(31_k), $k=1,2$, in the strip $\Pi L(i,l)$, $i\in I_l$, $l\in N$, that the partial derivatives $\partial\ln\psi_{i,l}(t)/\partial t_k$, $k=1,2$, are bounded; more precisely,

$$|\partial\ln\psi_{i,l}(t)/\partial t_k|\le 5,\qquad k=1,2.$$

Therefore, by (13_2), we have proved the boundedness of the partial derivatives $\partial\ln\tilde\psi(t)/\partial t_k$, $k=1,2$, in the "basic" strips $\Pi(i,l)$, $i\in I_l$, $l\in N$.

By taking account of the definition (13_3) of the function $\ln\tilde\psi(t)$ in the "transition" strip $\tilde\Pi(i{+}1,l)$, $i=0,1,\dots,l\times 2^l-1$, $l\in N$, we prove the boundedness of the products

$$[\ln\psi_{i+1,l}(t)-\ln\psi_{i,l}(t)](\partial/\partial t_k)\,e_{01}\left(\ln\sqrt{t_1+t_2};\ln\sqrt{\alpha_{i+1,l}},\ln\sqrt{\beta_{i+1,l}}\,\right),\qquad k=1,2.$$

Let us prove the boundedness of some of these products, say,

$$\ln\psi_{i,l}(t)(\partial/\partial t_k)\,e_{01}\left(\ln\sqrt{t_1+t_2};\ln\sqrt{\alpha_{i+1,l}},\ln\sqrt{\beta_{i+1,l}}\,\right),\qquad k=1,2;$$

the boundedness of the other product can be proved in a similar way. From (6), (24), and (10_2), we obtain the estimates

$$\left|\ln\psi_{i,l}(t)\frac{\partial}{\partial t_k}e_{01}\left(\ln\sqrt{t_1+t_2};\ln\sqrt{\alpha_{i+1,l}},\ln\sqrt{\beta_{i+1,l}}\right)\right|$$
$$\le\left(\left|(\Delta(i,l),\ln t)\right|+\left(\ln^2 t_1+\ln^2 t_2\right)\right)\times 2\exp\left[2\left(\ln\sqrt{\frac{\beta_{i+1,l}}{\alpha_{i+1,l}}}\right)^{-2}\right]\frac{1}{2\,(t_1+t_2)}$$
$$\le\exp\left[2\left(\ln\sqrt{2}\right)^{-2}\right]\left(\|\Delta(i,l)\|(\ln t_1+\ln t_2)+\left(\ln^2 t_1+\ln^2 t_2\right)\right)\frac{1}{t_1+t_2}\le 1,\qquad k=1,2.$$

Since the derivatives $\partial\ln\psi_{i,l}(t)/\partial t_k$ and $\partial\ln\psi_{i+1,l}(t)/\partial t_k$ are bounded in the "transition" strip $\tilde{\Pi}(i+1,l)$, $i=0,1,\ldots,l\times 2^l-1$, $l\in N$, it follows from definition (13_3) of the function $\ln\tilde{\psi}(t)$ that the partial derivatives $\partial\ln\tilde{\psi}(t)/\partial t_k$, $k=1,2$, are bounded in the "transition" strip $\tilde{\Pi}(i+1,l)$, $i=0,1,\ldots,l\times 2^l-1$, $l\in N$, as well. The boundedness of the derivatives $\partial\ln\tilde{\psi}(t)/\partial t_k$, $k=1,2$, in the "global transition" strips $\tilde{\Pi}(0,l+1)$, $l\in N$, and in the strip $\tilde{\Pi}(0,1)$ can be proved in a similar way.

We have thereby completely proved the boundedness of the partial derivatives $\partial\ln\tilde{\psi}(t)/\partial t_k$, $k=1,2$, in cases (a) and (b), in which the curve D is unbounded on the left and right, and the boundedness of the partial derivatives $\partial\ln\psi(t)/\partial t_k$, $k=1,2$, in the entire quadrant $R^2_{>1}$, where equation (1_1) constructed above is defined.

Let us now proceed to the proof of the boundedness of the partial derivatives $\partial\ln\psi(t)/\partial t_k$, $k=1,2$, in the quadrant $R^2_{>1}$ in case (c), in which the curve D is bounded on the left by a finite point. Obviously, the estimates

$$\begin{aligned}&\left|(\Delta(0,0),\ln t)\frac{\partial}{\partial t_1}e_{01}\left(\frac{\ln t_2}{\ln t_1};\frac{3}{|k(0,0)|},\frac{3}{|k(0,0)|}+\frac{1}{2}\right)\right|\\&\qquad\le 2e^8\|\Delta(0,0)\|(\ln t_1+\ln t_2)\,\frac{\ln t_2}{t_1\ln^2 t_1}\\&\qquad\le 2e^8\|\Delta(0,0)\|\left(3/|k(0,0)|+1/2+(3/|k(0,0)|+1/2)^2\right),\end{aligned}\tag{32_1}$$

$$\begin{aligned}&\left|(\Delta(0,0),\ln t)\frac{\partial}{\partial t_2}e_{01}\left(\frac{\ln t_2}{\ln t_1};\frac{3}{|k(0,0)|},\frac{3}{|k(0,0)|}+\frac{1}{2}\right)\right|\\&\qquad\le 2e^8\|\Delta(0,0)\|(\ln t_1+\ln t_2)/(t_2\ln t_1)\\&\qquad\le 2e^8\|\Delta(0,0)\|(3/|k(0,0)|+3/2)\end{aligned}\tag{32_2}$$

are valid in the quadrant $R^2_{>1}$. In view of definition (14) of the function $\ln\psi(t)$, to prove the boundedness of the partial derivatives $\partial\ln\psi(t)/\partial t_k$, $k=1,2$, in the quadrant $R^2_{>1}$, we need to justify the boundedness of the products $\ln\tilde{\psi}(t)(\partial/\partial t_k)\,e_{01}\,(\ln t_2/\ln t_1;3/|k(0,0)|,3/|k(0,0)|+1/2)$ in this quadrant. From (6), in each strip $\Pi L(i,l)$, $i\in I_l$, $l\in N$, we obtain the estimate

$$\left|\ln\psi_{i,l}(t)\frac{\partial}{\partial t_1}e_{01}\left(\frac{\ln t_2}{\ln t_1};\frac{3}{|k(0,0)|},\frac{3}{|k(0,0)|}+\frac{1}{2}\right)\right|$$
$$\le 2e^8\frac{\ln t_2}{t_1\ln^2 t_1}\Bigg(\|\Delta(i,l)\|(\ln t_1+\ln t_2)$$
$$\times e_{0110}\left(\frac{\ln t_2}{\ln t_1};\Theta_{i,l}-\tau_l-\frac{1}{4},\Theta_{i,l}-\tau_l,\Theta_{i,l}+\tau_l,\Theta_{i,l}+\tau_l+\frac{1}{4}\right)+\ln^2 t_1+\ln^2 t_2\Bigg)$$
$$\le 2e^8\Big((\Delta(l)/\nu_l)\left((\ln t_2)/\ln t_1+((\ln t_2)/\ln t_1)^2\right)+((\ln t_2)/t_1)\Big(1+((\ln t_2)/\ln t_1)^2\Big)\Big)$$
$$\le 2e^8\left(3/|k(0,0)|+1/2+(3/|k(0,0)|+1/2)^2+(3/|k(0,0)|+1/2)\left(1+(3/|k(0,0)|+1/2)^2\right)\right),$$

$$
\begin{aligned}
&\left|\ln\psi_{i,l}(t)\frac{\partial}{\partial t_2}e_{01}\left(\frac{\ln t_2}{\ln t_1};\frac{3}{|k(0,0)|},\frac{3}{|k(0,0)|}+\frac{1}{2}\right)\right| \\
&\quad\le\frac{2e^8}{t_2\ln t_1}\Biggl(\|\Delta(i,l)\|(\ln t_1+\ln t_2) \\
&\qquad\times e_{0110}\left(\frac{\ln t_2}{\ln t_1};\Theta_{i,l}-\tau_l-\frac{1}{4},\Theta_{i,l}-\tau_l,\Theta_{i,l}+\tau_l,\Theta_{i,l}+\tau_l+\frac{1}{4}\right)+\ln^2 t_1+\ln^2 t_2\Biggr) \\
&\quad\le 2e^8\Bigl((\Delta(l)/\nu_l)(1+(\ln t_2)/\ln t_1)+((\ln t_1)/t_2)\Bigl(1+((\ln t_2)/\ln t_1)^2\Bigr)\Bigr) \\
&\quad\le 2e^8\bigl(3/2+3/|k(0,0)|+|k(0,0)|\bigl(1+(3/|k(0,0)|+1/2)^2\bigr)/3\bigr)\,,
\end{aligned}
$$

which, together with definitions (13_1)–(13_5) of the function $\ln\tilde{\psi}(t)$, implies that the products

$$\ln\tilde{\psi}(t)(\partial/\partial t_k)\,e_{01}\,(\ln t_2/\ln t_1;3/|k(0,0)|,3/|k(0,0)|+1/2)$$

are bounded in the quadrant $R^2_{>1}$. Since these products and the partial derivatives $\partial\ln\tilde{\psi}(t)/\partial t_k$, $k=1,2$, are bounded, it follows from the estimates (32_1), (32_2), and (14) that the partial derivatives $\partial\ln\psi(t)/\partial t_k$, $k=1,2$, of the function $\ln\psi(t)$ are bounded in the quadrant $R^2_{>1}$ in case (c) as well.

We have thereby proved that the coefficients of equation (1_1) are bounded in the quadrant $R^2_{>1}$ and the assertion of the theorem is valid.

The following assertion gives a complete description of the left boundary lower exponent set.

Theorem 2. *A set D is the left boundary lower exponent set $\underline{D}_x\,(p')$ of some nontrivial solution $x(t)$, whose lower characteristic set P_x consists of more than one point, of some completely integrable Pfaff system* (1) *with bounded continuously differentiable coefficients if and only if it is empty or can be represented in the form of a closed concave monotone decreasing right- and lower-unbounded curve on the two-dimensional plane such that the slope of any tangent is negative and is not less than -1.*

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