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ORDINARY DIFFERENTIAL EQUATIONS

A Joint Description of the Boundary Exponent Sets of a Solution of a Linear Pfaff System: I

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We consider the linear Pfaff system

$$\partial x/\partial t_i = A_i(t)x, \qquad x \in R^n, \qquad t = (t_1, t_2) \in R^2_{>1}, \qquad i = 1, 2, \qquad n \in N, \tag{1_n}$$

with bounded continuously differentiable matrices $A_i(t)$ satisfying the complete integrability condition [1, pp. 14–24; 2, pp. 16–26]

$$\partial A_1(t)/\partial t_2 + A_1(t)A_2(t) = \partial A_2(t)/\partial t_1 + A_2(t)A_1(t), \qquad t \in R^2_{>1}.$$

Let $\lambda = \lambda[x] \in R^2$ and $p = p[x] \in R^2$ be the characteristic vector [3] and the lower characteristic vector [4], respectively, of a nontrivial solution $x : R^2_{>1} \to R^n \backslash \{0\}$ of system (1_n). They are determined by the conditions

$$\bar{l}_x(\lambda) \equiv \overline{\lim_{t\to\infty}} \frac{\ln\|x(t)\| - (\lambda, t)}{\|t\|} = 0, \qquad \bar{l}_x(\lambda - \varepsilon e_i) > 0, \quad \forall \varepsilon > 0, \quad i = 1, 2, \tag{2}$$

$$\underline{l}_x(p) \equiv \underline{\lim}_{t\to\infty} \frac{\ln\|x(t)\| - (p, t)}{\|t\|} = 0, \qquad \underline{l}_x(p + \varepsilon e_i) < 0, \quad \forall \varepsilon > 0, \quad i = 1, 2, \tag{3}$$

where $e_i = (2 - i, i - 1)$; $\Lambda_x = \bigcup\{\lambda[x]\}$ and $P_x = \bigcup\{p[x]\}$ are the characteristic set [3] and the lower characteristic set [4] of this solution. By $\lambda', \lambda'' \in \Lambda_x$ $(\lambda'_1 \le \lambda_1 \le \lambda''_1, \forall \lambda \in \Lambda_x)$ [respectively, $p', p'' \in P_x$ $(p'_1 \le p_1 \le p''_1, \forall p \in P_x)$] we denote the left and right boundary points, respectively, of the characteristic set Λ_x (respectively, the lower characteristic set P_x) of the solution $x(t)$.

On the basis of Demidovich's definition of the characteristic exponent [5] of a solution of an ordinary differential system, for a characteristic vector $\lambda \in \Lambda_x$ and a lower characteristic vector $p \in P_x$, one introduces the *upper characteristic exponent* [6] $\bar{d} = \bar{d}_x(\lambda) \in R^2$ and the *lower characteristic exponent* [7] $\underline{d} = \underline{d}_x(p) \in R^2$ of the solution x by the conditions

$$\overline{\ln}_x\left(\lambda, \bar{d}\right) \equiv \overline{\lim_{t\to\infty}} \frac{\ln\|x(t)\| - (\lambda, t) - \left(\bar{d}, \ln t\right)}{\|\ln t\|} = 0, \quad \overline{\ln}_x\left(\lambda, \bar{d} - \varepsilon e_i\right) > 0, \quad \forall \varepsilon > 0, \quad i = 1, 2, \tag{4}$$

$$\underline{\ln}_x(p, \underline{d}) \equiv \underline{\lim}_{t\to\infty} \frac{\ln\|x(t)\| - (p, t) - (\underline{d}, \ln t)}{\|\ln t\|} = 0, \quad \underline{\ln}_x(p, \underline{d} + \varepsilon e_i) < 0, \quad \forall \varepsilon > 0, \quad i = 1, 2, \tag{5}$$

where $\ln t \equiv (\ln t_1, \ln t_2) \in R^2_+$; then one defines the *upper exponent set* $\bar{D}_x(\lambda) = \bigcup\left\{\bar{d}_x(\lambda)\right\}$ and the *lower exponent set* $\underline{D}_x(p) = \bigcup\left\{\underline{d}_x(p)\right\}$. The sets $\bar{D}_x(\lambda')$ and $\bar{D}_x(\lambda'')$ [respectively, $\underline{D}_x(p')$ and $\underline{D}_x(p'')$] are referred [8] to as the *left* and *right boundary upper* (respectively, *lower*) *exponent sets* of the solution $x(t)$.

Necessary properties of the characteristic set Λ_x of a solution $x(t) \ne 0$ of system (1_n) were obtained in [3], and their sufficiency was proved in [9]; the characteristic set of any nontrivial solution of system (1_n) was thereby completely described. Necessary properties of the lower characteristic set P_x of a solution $x(t) \ne 0$ and necessary conditions on the mutual arrangement of the characteristic and lower characteristic sets were given in [4]; the problem of simultaneous realization of two arbitrary given sets satisfying necessary conditions by the characteristic and lower characteristic sets of a nontrivial solution $x : R^2_{>1} \to R\backslash\{0\}$ of some Pfaff equation

$$\partial x/\partial t_i = a_i(t)x, \qquad x \in R, \qquad t \in R^2_{>1}, \qquad i = 1, 2, \tag{1_1}$$

was solved in [10].

Krupchik [7] completely described arbitrary nonempty interior lower and upper exponent sets $\underline{D}_x(p)$, $p \neq p', p''$, and $\bar{D}_x(\lambda)$, $\lambda \neq \lambda', \lambda''$, of a nontrivial solution $x(t)$ of system (1_n). These sets are the lines $d_1 + d_2 = \underline{c}_x(p)$ and $d_1 + d_2 = \bar{c}_x(\lambda)$, respectively. Nonempty boundary exponent sets have a much more complicated structure. The following necessary properties [11] of boundary exponent sets are known for the nontrivial cases in which the lower characteristic set P_x and the characteristic set Λ_x consist of more than a single point: (1) the nonempty left and right boundary lower exponent sets $\underline{D}_x(p')$ and $\underline{D}_x(p'')$ of a solution $x(t) \neq 0$ of system (1_n) are closed concave monotone decreasing curves on the two-dimensional plane; they are unbounded on the right and below (respectively, on the left and above), and their slope at each interior point is not less (respectively, greater) than -1; (2) the nonempty left and right boundary upper exponent sets $\bar{D}_x(\lambda')$ and $\bar{D}_x(\lambda'')$ are closed convex monotone decreasing curves on the two-dimensional plane; they are unbounded on the right and below (respectively, on the left and above), and their slope at each interior point is not greater (respectively, less) than -1. Moreover, for each curve on the two-dimensional plane satisfying the necessary properties of left boundary lower exponent sets, an integrable Pfaff equation (1_1) with infinitely differentiable bounded coefficients such that the left boundary lower exponent set of its arbitrary nontrivial solution coincides with this curve was constructed in [8]. In this connection, by analogy with the case of characteristic sets, we encounter the problem of simultaneous realization of four arbitrarily given sets satisfying only the necessary conditions as the exponent sets of a single nontrivial solution of some system (1_n). This problem is solved in the following assertion.

Theorem 1. *Let the following objects be given*:

an arbitrary positive integer n;

arbitrary closed concave monotone decreasing curves $D^{(1)}$ *and* $D^{(2)}$ *on the two-dimensional plane unbounded on the right and below and on the left and above, respectively, and* $D^{(2)}$ *on a two-dimensional plane with the angular coefficient of any tangent and having slopes not less* (*respectively, greater*) *than* -1 *at each interior point*;

arbitrary closed convex monotone decreasing curves $D^{(3)}$ *and* $D^{(4)}$ *on the two-dimensional plane unbounded on the right and below and on the left and above, respectively, and* $D^{(4)}$ *on a two-dimensional plane with the angular coefficient of any tangent and having slopes not greater* (*respectively, less*) *than* -1 *at each interior point.*

Then there exists a completely integrable Pfaff system (1_n) *with infinitely differentiable bounded coefficients such that its arbitrary nontrivial solution* $x : R^2_{>1} \to R^n\backslash\{0\}$ *has the left and right boundary lower exponent sets* $\underline{D}_x(p') = D^{(1)}$ *and* $\underline{D}_x(p'') = D^{(2)}$, *respectively, and the left and right boundary upper exponent sets* $\bar{D}_x(\lambda') = D^{(3)}$ *and* $\bar{D}_x(\lambda'') = D^{(4)}$, *respectively.*

Proof of Theorem 1. First, we note that it suffices to consider the case $n = 1$. Indeed, after the construction of a one-dimensional completely integrable Pfaff equation (1_1) with infinitely bounded differentiable coefficients $a_i(t)$, $i = 1, 2$, and with the desired boundary exponent sets of a nontrivial solution $x_1 : R^2_{>1} \to R\backslash\{0\}$, as the desired n-dimensional system (1_n) one can choose the diagonal system with coefficient matrix $A_i(t) = \operatorname{diag}[a_i(t), \ldots, a_i(t)]$, $i = 1, 2$, of order n. Then the matrix $X(t) = x_1(t)E_n$, where E_n is the identity matrix of order n, is the principal solution matrix of system (1_n), and an arbitrary nontrivial solution $x : R^2_{>1} \to R^n\backslash\{0\}$ of the latter can be represented in the form $x(t) = X(t)c = x_1(t)c$ with some $c \in R^n\backslash\{0\}$. Therefore, the characteristic and lower characteristic vectors and the upper and lower characteristic exponents of this solution are the same as for the solution $x_1(t)$ of Eq. (1_1): $\lambda[x] = \lambda[x_1]$, $p[x] = p[x_1]$, $\bar{d}_x(\lambda[x]) = \bar{d}_{x_1}(\lambda[x_1])$, and $\underline{d}_x(p[x]) = \underline{d}_{x_1}(p[x_1])$; consequently, the corresponding characteristic and exponent sets also coincide.

We construct the desired equation (1_1) by constructing a nontrivial solution.

1. PARTITIONS OF THE CURVES $D^{(q)}$, $q = 1, 2, 3, 4$

It follows from the properties of the curves $D^{(q)}$, $q = 1, 2, 3, 4$, that each of them necessarily has one of the following three forms. The curve $D^{(1)}$ can be

(1_1) unbounded on the left and bounded above;

(2_1) unbounded on the left and above;

(3_1) bounded on the left and above.

The curve $D^{(2)}$ can be

(1_2) unbounded below and bounded on the right;

(2_2) unbounded below and on the right;

(3_2) bounded below and on the right.

The curve $D^{(3)}$ can be

(1_3) unbounded above and bounded on the left;

(2_3) unbounded above and on the left;

(3_3) bounded above and on the left.

The curve $D^{(4)}$ can be

(1_4) unbounded on the right and bounded below;

(2_4) unbounded on the right and below;

(3_4) bounded on the right and below.

To construct a solution $x(t)$ simultaneously realizing the left and right boundary lower exponent sets $\underline{D}_x(p') = D^{(1)}$ and $\underline{D}_x(p'') = D^{(2)}$ as well as the left and right boundary upper exponent sets $\bar{D}_x(\lambda') = D^{(3)}$ and $\bar{D}_x(\lambda'') = D^{(4)}$, we perform the following partition of the curves $D^{(q)}$, $q = 1, 2, 3, 4$. We take a number $\gamma > 0$.

1.1. Partitions of the Curves $D^{(q)}$, $q = 1, 4$

Following [8], in cases (1_q) and (2_q) of the curve $D^{(q)}$, $q = 1, 4$, unbounded on the left and on the right, we construct its lth partition $D_l^{(q)} = \bigcup_{i=1}^{l\times 2^l} \left\{\Delta^{(q)}(i,l)\right\} \subset D^{(q)}$ using the points $\Delta^{(q)}(i,l) \in D^{(q)}$ with first components $\Delta_1^{(q)}(i,l) = \left(i \times 2^{1-l} - l\right)\gamma$, $i \in \left\{1, \ldots, l \times 2^l\right\} \equiv I_l$, $l \in N$; and in case (3_q) of the curve $D^{(q)}$ with the left boundary point (for $q = 1$) or the right boundary point (for $q = 4$)[1] $\Delta^{(q)}(0,0) \in D^{(q)}$, we construct the partition $D_l^{(q)}$ of this curve using the points $\Delta^{(q)}(i,l) \in D^{(q)}$ with first components $\Delta_1^{(q)}(i,l) = \Delta_1^{(q)}(0,0) - (-1)^q i\gamma \times 2^{-l}$, $i \in I_l$, $l \in N$. By infinitely continuing the partition of the curve $D^{(q)}$, we obtain the countable set $D_\infty^{(q)} = \bigcup_{l=1}^{+\infty} \bigcup_{i=1}^{l\times 2^l} \left\{\Delta^{(q)}(i,l)\right\} \subset D^{(q)}$, which is everywhere dense on the curve $D^{(q)}$.

1.2. Partitions of the Curves $D^{(q)}$, $q = 2, 3$

In cases (1_q) and (2_q) of the curve $D^{(q)}$, $q = 2, 3$, unbounded below and above, its lth partition $D_l^{(q)}$, $l \in N$, consists of the points $\Delta^{(q)}(i,l) \in D^{(q)}$ with second components $\Delta_2^{(q)}(i,l) = \left(i \times 2^{1-l} - l\right)\gamma$, $i \in I_l$, and in case (3_q) of the curve $D^{(q)}$ with the right boundary point (for $q = 2$) or the left boundary point (for $q = 3$) $\Delta^{(q)}(0,0) \in D^{(q)}$, it consists of the points $\Delta^{(q)}(i,l) \in D^{(q)}$ with the second components $\Delta_2^{(q)}(i,l) = \Delta_2^{(q)}(0,0) + (-1)^q i\gamma \times 2^{-l}$, $i \in I_l$. As a result, we obtain a countable set $D_\infty^{(q)}$, which is everywhere dense on the curve $D^{(q)}$.

By $D^{(q)}(l)$, $q = 1, 2, 3, 4$, we denote the part of the curve $D^{(q)}$ lying between the points $\Delta^{(q)}(1,l)$ from $D_l^{(q)}$ and $\Delta^{(q)}\left(l \times 2^l, l\right)$ from $D_l^{(q)}$, including these points.

2. CONSTRUCTION OF THE SOLUTION

We define the desired solution $x(t)$ by the relation $x(t) = \varphi(t)\psi(t)$. We construct the function $\varphi(t)$ so as to ensure that its characteristic set Λ_φ and the lower characteristic set P_φ are

[1] In the sense of the above-given definition.

nontrivial, i.e., other than singletons. We define the function $\psi(t)$ so as to ensure that the characteristic and lower characteristic sets of the resulting solution $x(t)$ coincide with the corresponding sets of the function $\varphi(t)$ and the solution $x(t)$ has the left and right boundary lower exponent sets $\underline{D}_x(p') = D^{(1)}$ and $\underline{D}_x(p'') = D^{(2)}$ and the left and right boundary upper exponent sets $\bar{D}_x(\lambda') = D^{(3)}$ and $\bar{D}_x(\lambda'') = D^{(4)}$.

To glue together various infinitely differentiable functions with preservation of smoothness, we also use the infinitely differentiable functions

$$e_{101}(\tau;\alpha_1,\alpha_2,\alpha_3) = e_{01}(\tau;\alpha_2,\alpha_3) + [1 - e_{01}(\tau;\alpha_1,\alpha_2)], \tag{6}$$

$$e_{0110}(\tau;\alpha_1,\alpha_2,\alpha_3,\alpha_4) = e_{01}(\tau;\alpha_1,\alpha_2)(1 - e_{01}(\tau;\alpha_3,\alpha_4)), \tag{7}$$

$$\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4, \quad \tau \in R,$$

defined on the basis of the function [12, p. 54 of the Russian translation]

$$e_{01}(\tau;\tau_1,\tau_2) = \begin{cases} \exp\left\{-(\tau-\tau_1)^{-2}\exp\left[-(\tau_2-\tau)^{-2}\right]\right\} & \text{for} \quad \tau \in (\tau_1,\tau_2) \\ \left[1 + \operatorname{sgn}\left(\tau - 2^{-1}(\tau_1+\tau_2)\right)\right]/2 & \text{for} \quad \tau \notin (\tau_1,\tau_2), \end{cases}$$

where $-\infty < \tau_1 < \tau_2 < +\infty$. To prove the boundedness of the coefficients of Eq. (1_1) to be constructed, we need the following assertion.

Lemma. *The derivative of the function $e_{01}(\tau;\tau_1,\tau_2)$ satisfies the estimate*

$$0 \le \frac{de_{01}(\tau;\tau_1,\tau_2)}{d\tau} \le \begin{cases} 2\exp\left[2(\tau_2-\tau_1)^{-2}\right] & \text{if} \quad \tau_2-\tau_1 \le 1/2 \\ 4 & \text{if} \quad \tau_2-\tau_1 \ge 2 \end{cases} \tag{L_1}$$

on the interval $[\tau_1,\tau_2]$.

Proof of the lemma. We write $m(\Theta) \equiv \exp(-\Theta^{-2})$, $\Theta > 0$; then for the derivative in question, we have the representation

$$\begin{aligned} de_{01}(\tau;\tau_1,\tau_2)/d\tau &= 2m(\tau_2-\tau)\left[(\tau-\tau_1)^{-3}e_{01}(\tau;\tau_1,\tau_2)\right] \\ &\quad + 2\left[(\tau_2-\tau)^{-3}m(\tau_2-\tau)\right]\left[(\tau-\tau_1)^{-2}e_{01}(\tau;\tau_1,\tau_2)\right] \\ &\equiv 2m(\tau_2-\tau)s_1(\tau) + 2s_2(\tau)s_3(\tau), \quad \tau \in (\tau_1,\tau_2). \end{aligned} \tag{L_2}$$

Let us first consider the second case, in which $\tau_2 - \tau_1 \ge 2$. We compute the maximum value of the function

$$g_k(\tau;\gamma) \equiv (\tau-\tau_1)^{-k}\exp\left[-\gamma(\tau-\tau_1)^{-2}\right], \qquad k \in \{2,3\},$$

with some parameter $\gamma \in (0,1)$ on the interval $(\tau_1,+\infty)$. This positive function increases on the interval $(\tau_1,\eta_k]$, $\eta_k = \tau_1 + \sqrt{2\gamma/k} \in (\tau_1,\tau_1+1)$, up to its maximum value $g_k(\eta_k;\gamma) = \left(\sqrt{k/(2e\gamma)}\right)^k$ and decays on the interval $(\eta_k,+\infty)$. Let us estimate the derivative (L_2) on the interval $(\tau_1,\tau_1+1]$. We have the inequalities

$$m(\tau_2-\tau) \le m(\eta), \qquad s_1(\tau) \le g_3(\eta_3;m(\eta-1)) = \left(\sqrt{3/(2e)}\right)^3\exp\left[3(\eta-1)^{-2}/2\right],$$

$$s_2(\tau) \le s_2(\tau_1+1) \le m(\eta-1), \qquad s_3(\tau) \le g_2(\eta_2;m(\eta-1)) = [em(\eta-1)]^{-1},$$

$$\eta \equiv \tau_2-\tau_1, \quad \tau \in (\tau_1,\tau_1+1].$$

By taking account of the representation (L_2) with regard to these inequalities, the obvious inequality $\sqrt{e} > 3/2$, and the growth of the function $f(\eta) \equiv 2^{-1}(\eta^2+4\eta-2)(\eta^2-\eta)^{-2}$ on the interval $[2,+\infty)$, we obtain the estimates

$$\begin{aligned} de_{01}(\tau;\tau_1,\tau_2)/d\tau &\le \left(\sqrt{2}\right)^{-1}(3/e)^{3/2}\exp(f(\eta)) + 2e^{-1} \\ &\le 3\sqrt{3/(2\sqrt{e})} + 2e^{-1} < 3+1 = 4, \qquad \tau \in (\tau_1,\tau_1+1]. \end{aligned}$$

On the interval $[\tau_1+1,\tau_2-1]$, the functions $e_{01}(\tau;\tau_1,\tau_2)$ and $s_2(\tau)$ are increasing, and the functions $m(\tau_2-\tau)$ and $(\tau-\tau_1)^{-k}$, $k\in\{2,3\}$, are decreasing. Therefore, on this interval, we have the inequalities

$$\begin{gathered} m(\tau_2-\tau)\le m(\eta-1), \qquad (\tau-\tau_1)^{-k}\le 1, \qquad k\in\{2,3\},\\ e_{01}(\tau;\tau_1,\tau_2)\le e_{01}(\tau_2-1;\tau_1,\tau_2)=m\left(\sqrt{e}(\eta-1)\right), \qquad s_2(\tau)\le s_2(\tau_2-1)=e^{-1},\\ s_i(\tau)\le e_{01}(\tau;\tau_1,\tau_2)\le m\left(\sqrt{e}(\eta-1)\right), \qquad i=1,3, \qquad \tau\in[\tau_1+1,\tau_2-1], \end{gathered}$$

which imply the estimates

$$de_{01}(\tau;\tau_1,\tau_2)/d\tau\le 2m\left(\sqrt{e}(\eta-1)\right)\left[m(\eta-1)+e^{-1}\right]\le 4, \qquad \tau\in[\tau_1+1,\tau_2-1].$$

Finally, let us consider the remaining interval (τ_2-1,τ_2). Since the functions $m(\tau_2-\tau)$ and $(\tau-\tau_1)^{-k}$, $k\in\{2,3\}$, are decreasing on this interval, we have the inequalities

$$\begin{gathered} m(\tau_2-\tau)\le m(1)=e^{-1}, \qquad e_{01}(\tau;\tau_1,\tau_2)\le 1, \qquad (\tau-\tau_1)^{-k}\le(\eta-1)^{-k}\le 1,\\ s_2(\tau)\le s_2\left(\tau_2-\sqrt{2/3}\right)=\left(\sqrt{3/(2e)}\right)^3, \qquad \tau\in(\tau_2-1,\tau_2), \end{gathered}$$

and hence the estimates

$$de_{01}(\tau;\tau_1,\tau_2)/d\tau\le\left(2+3\sqrt{3/(2e)}\right)e^{-1}<2, \qquad \tau\in(\tau_2-1,\tau_2).$$

To prove the estimate (L_1) for $\tau_2-\tau_1\le 1/2$, we use the inequalities

$$\begin{gathered} (\tau-\tau_1)^{-m}e_{01}(\tau;\tau_1,\tau_2)\le\left[\sqrt{m/(2e)}\exp(\tau_2-\tau_1)^{-2}\right]^m, \quad \tau\in(\tau_1,\tau_2), \quad m\in N,\\ (\tau_2-\tau)^{-m}\exp\left[-(\tau_2-\tau)^{-2}\right]\le\left(\sqrt{m/(2e)}\right)^m, \quad \tau\in(\tau_1,\tau_2), \quad m\in N, \end{gathered}$$

obtained in a lemma in [13]. Then we have the estimates

$$\begin{aligned} &de_{01}(\tau;\tau_1,\tau_2)/d\tau\\ &\le 2\left(\left[\sqrt{3/(2e)}\exp(\tau_2-\tau_1)^{-2}\right]^3\exp\left(-(\tau_2-\tau_1)^{-2}\right)+\left[\sqrt{1/e}\exp(\tau_2-\tau_1)^{-2}\right]^2\left(\sqrt{3/(2e)}\right)^3\right)\\ &=2\left(\exp\left[2(\tau_2-\tau_1)^{-2}\right]\right)\left(\sqrt{3/(2e)}\right)^3(1+1/e)\le 2\exp\left[2(\tau_2-\tau_1)^{-2}\right]. \end{aligned}$$

The proof of the lemma is complete.

Proof of Theorem 1 (continuation). Let $d_2-\Delta_2^{(q)}(i,l)=k^{(q)}(i,l)\left(d_1-\Delta_1^{(q)}(i,l)\right)$, $d\in R^2$, where $k^{(q)}(i,l)\in[-1,0)$, for $q=1,4$ and $k^{(q)}(i,l)\in(-\infty,-1]$ for $q=2,3$, be the equation of the tangent to the curve $D^{(q)}$, $q=1,2,3,4$, at the point $\Delta^{(q)}(i,l)\in D_l^{(q)}\subset D^{(q)}$, $i\in I_l$, $l\in N$, lying not below (for $q=1,2$) and not above (for $q=3,4$) the curve $D^{(q)}$. The existence of such tangents follows from the convexity or concavity of the corresponding curves and from the fact that the partition $D_l^{(q)}$ does not contain the boundary point $\Delta^{(q)}(0,0)$ at which the slope of the curve $D^{(q)}$ can vanish for $q=1,4$ and can be equal to $-\infty$ for $q=2,3$. Moreover, if a point has been used in the partition, then for all subsequent partitions, we draw the same tangent at this point. We also set

$$\begin{aligned} \Theta_{i,l}^{(q)}&=1/\left|k^{(q)}(i,l)\right|, \qquad i\in I_l, \qquad l\in N, \qquad q=1,2,3,4,\\ \Theta_l^{(q)}&=\max\left\{\Theta_{i,l}^{(q)}:\ \Theta_{i,l}^{(q)}\ne 1,\ i\in I_l\right\},\\ \Omega_l^{(q)}&=\min\left\{\Theta_{i,l}^{(q)}:\ \Theta_{i,l}^{(q)}\ne 1,\ i\in I_l\right\}, \qquad q=1,2,3,4, \qquad l\in N,\\ \Delta_1(l)&=\max\left\{\left\|\Delta^{(q)}(i,l)\right\|:\ i\in I_l,\ q=1,2,3,4\right\},\\ \Delta_2(l)&=\min\left\{2^{-l}\left\|\Delta^{(q)}\left(l\times 2^l,l\right)-\Delta^{(q)}(1,l)\right\|^{-1}:\ q=1,2,3,4\right\}, \qquad l\in N. \end{aligned}$$

For each $l \in N$ and $i \in I_l$, we introduce functions $\psi_{i,l}^{(q)}(t)$, $q = 1,2,3,4$, to be used in the realization of the left and right boundary lower exponent sets $\underline{D}_x(p') = D^{(1)}$ and $\underline{D}_x(p'') = D^{(2)}$ and the left and right boundary upper exponent sets $\bar{D}_x(\lambda') = D^{(3)}$ and $\bar{D}_x(\lambda'') = D^{(4)}$. We also introduce sectors $S^{(q)}(i,l)$, $q = 1,2,3,4$, of the quadrant $R^2_{>1}$. If $q = 1,4$ and the slope $k^{(q)}(i,l)$ of the curve $D^{(q)}$ at the point $\Delta^{(q)}(i,l) \in D_l^{(q)} \subset D^{(q)}$ is different from -1, then, following [8], we define the function $\psi_{i,l}^{(q)}(t)$ by the relation

$$\begin{aligned} \ln \psi_{i,l}^{(q)}(t) \equiv \left(\Delta^{(q)}(i,l), \ln t\right) e_{0110}\left(\frac{\ln t_2}{\ln t_1}; \Theta_{i,l}^{(q)} - \tau_l^{(q)} - \frac{1}{4}, \Theta_{i,l}^{(q)} - \tau_l^{(q)}, \Theta_{i,l}^{(q)} + \tau_l^{(q)}, \Theta_{i,l}^{(q)} + \tau_l^{(q)} + \frac{1}{4}\right) \\ - (-1)^q \|\ln t\|^2 e_{101}\left(\frac{\ln t_2}{\ln t_1}; \Theta_{i,l}^{(q)} - \tau_l^{(q)}, \Theta_{i,l}^{(q)}, \Theta_{i,l}^{(q)} + \tau_l^{(q)}\right), \quad t \in R^2_{>1}, \quad q = 1,4, \end{aligned} \qquad (8_{1,1})$$

$$\Theta_{i,l}^{(q)} \in (1, +\infty), \quad \tau_l^{(q)} \equiv \min\left\{1/2; \Delta_2(l); \left(\Omega_l^{(q)} - 1\right)/2\right\}, \qquad (9_1)$$

and the sector $S^{(q)}(i,l)$ corresponding to the point $\Delta^{(q)}(i,l) \in D^{(q)}$ by the formula

$$S^{(q)}(i,l) \equiv \left\{t \in R^2_{>1} : \ \left|\ln t_2/\ln t_1 - \Theta_{i,l}^{(q)}\right| < \tau_l^{(q)}\right\};$$

otherwise, we set

$$\begin{aligned} \ln \psi_{i,l}^{(q)}(t) \equiv \left(\Delta^{(q)}(i,l), \ln t\right) e_{0110}\left(t_2/t_1; e-2, e, 3e, 4e\right) \\ - (-1)^q \|\ln t\|^2 e_{101}\left(t_2/t_1; e, 2e, 3e\right), \quad t \in R^2_{>1}, \end{aligned} \qquad (8_{1,2})$$

$S^{(q)}(i,l) \equiv \{t \in R^2_{>1} : \ e < t_2/t_1 < 3e\}$, $q = 1,4$. If $q = 2,3$, and the slope $k^{(q)}(i,l)$ at the point $\Delta^{(q)}(i,l) \in D_l^{(q)} \in D^{(q)}$ is different from -1, then we define the function $\psi_{i,l}^{(q)}(t)$ by the formula

$$\begin{aligned} \ln \psi_{i,l}^{(q)}(t) \equiv \left(\Delta^{(q)}(i,l), \ln t\right) \\ \times e_{0110}\left(\frac{\ln t_2}{\ln t_1}; \Theta_{i,l}^{(q)} - \tau_l^{(q)} - \frac{\Omega_l^{(q)}}{4}, \Theta_{i,l}^{(q)} - \tau_l^{(q)}, \Theta_{i,l}^{(q)} + \tau_l^{(q)}, \Theta_{i,l}^{(q)} + \tau_l^{(q)} + \frac{\Omega_l^{(q)}}{4}\right) \\ + (-1)^q \|\ln t\|^2 e_{101}\left(\frac{\ln t_2}{\ln t_1}; \Theta_{i,l}^{(q)} - \tau_l^{(q)}, \Theta_{i,l}^{(q)}, \Theta_{i,l}^{(q)} + \tau_l^{(q)}\right), \quad t \in R^2_{>1}, \quad q = 2,3, \end{aligned} \qquad (8_{2,1})$$

$$\Theta_{i,l}^{(q)} \in (0,1), \qquad \tau_l^{(q)} \equiv \min\left\{\Omega_l^{(q)}/2; \Delta_2(l); \left(1 - \Theta_l^{(q)}\right)/2\right\}, \qquad (9_2)$$

and set $S^{(q)}(i,l) \equiv \left\{t \in R^2_{>1} : \ \left|\ln t_2/\ln t_1 - \Theta_{i,l}^{(q)}\right| < \tau_l^{(q)}\right\}$. If $k^{(q)}(i,l) = -1$, then we define the function $\psi_{i,l}^{(q)}(t)$ by the formula

$$\begin{aligned} \ln \psi_{i,l}^{(q)}(t) \equiv \left(\Delta^{(q)}(i,l), \ln t\right) e_{0110}\left(t_2/t_1; 1/(4e), 1/(3e), 1/e, 1/e + 1/2\right) \\ + (-1)^q \|\ln t\|^2 e_{101}\left(t_2/t_1; 1/(3e), 1/(2e), 1/e\right), \quad t \in R^2_{>1}, \end{aligned} \qquad (8_{2,2})$$

and set $S^{(q)}(i,l) \equiv \{t \in R^2_{>1} : \ 1/(3e) < t_2/t_1 < 1/e\}$.

We take an arbitrary number $l \in N$. Since the sets $D^{(q)}(l)$, $q = 1,2,3,4$, are bounded, it follows that there exists a number $c(l) > 0$ such that $\|d\| \le c(l)$ for all $d \in \bigcup_{q=1}^4 D^{(q)}(l) \equiv D(l)$. By using these inequalities, for the functions $\psi_{i,l}^{(q)}(t)$, $i \in I_l$, $q = 1,2,3,4$, and $d \in D(l)$, we obtain the estimates $\ln \psi_{i,l}^{(q)}(t) - (d, \ln t) \ge \|\ln t\|^2 - \left(\left\|\Delta^{(q)}(i,l)\right\| + \|d\|\right)\|\ln t\| \ge \|\ln t\|^2 - 2c(l)\|\ln t\|$, $t \in R^2_{>1} \backslash S^{(q)}(i,l)$, $q = 1,2$, and

$$\ln \psi_{i,l}^{(q)}(t) - (d, \ln t) \le -\|\ln t\|^2 + \left(\left\|\Delta^{(q)}(i,l)\right\| + \|d\|\right)\|\ln t\| \le -\|\ln t\|^2 + 2c(l)\|\ln t\|,$$

$t \in R^2_{>1} \backslash S^{(q)}(i,l)$, $q = 3, 4$. These estimates imply that there exists a number $T_l \geq 1$ such that

$$\ln \psi^{(q)}_{i,l}(t) - (d, \ln t) \geq 0, \quad t \in R^2_{>1} \backslash S^{(q)}(i,l), \quad \|t\| \geq T_l, \quad \forall d \in D(l), \quad i \in I_l, \quad q = 1, 2, \quad (10_1)$$
$$\ln \psi^{(q)}_{i,l}(t) - (d, \ln t) \leq 0, \quad t \in R^2_{>1} \backslash S^{(q)}(i,l), \quad \|t\| \geq T_l, \quad \forall d \in D(l), \quad i \in I_l, \quad q = 3, 4. \quad (10_2)$$

By the definition of the sector $S^{(q)}(i,l)$, $i \in I_l$, $l \in N$, $q = 1, 2, 3, 4$, for the case in which $k^{(q)}(i,l) = -1$, we obtain the estimates $t_2 > et_1$, $t \in S^{(q)}(i,l)$, $q = 1, 4$, and $t_2 < t_1/e$, $t \in S^{(q)}(i,l)$, $q = 2, 3$. If $k^{(q)}(i,l) \neq -1$, then from the inequalities

$$\ln t_2/\ln t_1 \geq \Theta^{(q)}_{i,l} - \tau^{(q)}_l \geq \Omega^{(q)}_l - \left(\Omega^{(q)}_l - 1\right)/2 = \left(\Omega^{(q)}_l + 1\right)/2 > 1, \quad t \in S^{(q)}(i,l), \quad q = 1, 4,$$
$$\ln t_2/\ln t_1 \leq \Theta^{(q)}_{i,l} + \tau^{(q)}_l \leq \Theta^{(q)}_l + \left(1 - \Theta^{(q)}_l\right)/2 = \left(\Theta^{(q)}_l + 1\right)/2 < 1, \quad t \in S^{(q)}(i,l), \quad q = 2, 3,$$

we find that there exists a sufficiently large number $b_l > 0$ such that

$$t_2 \geq et_1, \qquad t \in S^{(q)}(i,l), \qquad t_1 + t_2 \equiv \zeta(t) \geq b_l, \qquad q = 1, 4, \qquad (11_1)$$
$$t_2 \leq t_1/e, \qquad t \in S^{(q)}(i,l), \qquad \zeta(t) \geq b_l, \qquad q = 2, 3. \qquad (11_2)$$

It follows from the definition of the sector $S^{(q)}(i,l)$, $q = 1, 4$, $i \in I_l$, $l \in N$, and from (9_1) that if $k^{(q)}(i,l) \neq -1$, then $\left|\ln t_2/\ln t_1 - \Theta^{(q)}_{i,l}\right| < \Delta_2(l)$ for all $t \in S^{(q)}(i,l)$. Let us show that if $k^{(q)}(i,l) = -1$, then the similar estimate $|\ln t_2/\ln t_1 - 1| \leq \Delta_2(l)$ is valid for all $t \in S^{(q)}(i,l)$ satisfying the condition $\zeta(t) \geq \exp\left(3\Delta_2^{-1}(l) + 3\right) \equiv h_l$. Since, in this case, the inequality $t_2 \geq et_1$ is valid for all $t \in S^{(q)}(i,l)$, we have $\ln t_2/\ln t_1 - 1 > 0$. On the other hand, the estimates

$$\ln t_2/\ln t_1 - 1 \leq 3/\ln t_1 \leq \Delta_2(l)$$

are valid in the sector $S^{(q)}(i,l)$. Therefore, the inequalities

$$\left|\ln t_2/\ln t_1 - \Theta^{(q)}_{i,l}\right| \leq \Delta_2(l), \qquad t \in S^{(q)}(i,l), \qquad \zeta(t) \geq h_l, \qquad q = 1, 4, \qquad (12_1)$$

are valid for all $i \in I_l$ and $l \in N$. In a similar way, one can obtain the estimates

$$\left|\ln t_2/\ln t_1 - \Theta^{(q)}_{i,l}\right| \leq \Delta_2(l), \qquad t \in S^{(q)}(i,l), \qquad \zeta(t) \geq h_l, \qquad q = 2, 3, \qquad (12_2)$$

for all $i \in I_l$ and $l \in N$.

For any $l \in N$, there exists a number $\varrho_l > 0$ such that if $\zeta(t) \geq \varrho_l$, then all directions in which the points $\Delta^{(q)}(i,l) \in D^{(q)}_l$, $i \in I_l$, are realized by the corresponding boundary exponent sets lie below the curve $t_2 = t_1^{\sqrt[3]{t_1}}$ for $q = 1, 4$ and above the curve $t_1 = t_2^{\sqrt[3]{t_2}}$ for $q = 2, 3$; more precisely,

$$t_1^{\sqrt[3]{t_1}} \geq \max\left\{2et_1, t_1^{\Theta^{(q)}_{i,l}}\right\}, \qquad \zeta(t) \geq \varrho_l, \qquad i \in I_l, \qquad q = 1, 4, \qquad (13_1)$$
$$t_2^{\sqrt[3]{t_2}} \geq \max\left\{2et_2, t_2^{1/\Theta^{(q)}_{i,l}}\right\}, \qquad \zeta(t) \geq \varrho_l, \qquad i \in I_l, \qquad q = 2, 3. \qquad (13_2)$$

We split the entire quadrant $R^2_{>1}$, that is, the domain of the solution $x(t)$ to be constructed, by lines of the form $\zeta(t) = \mathrm{const}$ into disjoint strips. By setting $R_l = T_l + h_l + b_l + \varrho_l$, $l \in N$, and by using some values $\eta^{(1)}_1 \geq R_1$, $c \geq \exp(100)$, for $l \in N$, we introduce the numbers

$$\nu^{(q)}_l = c\left(\Theta^{(q)}_l\right)^6 \left(\Delta_1^2(l) + \exp\left(4\left(\tau^{(q)}_l\right)^{-2}\right)\right), \qquad \mu^{(q)}_l = \left(\nu^{(q)}_l\right)^{4\Theta^{(q)}_l}, \qquad q = 1, 4,$$
$$\nu^{(q)}_l = c\left(\Delta_1^2(l) + 1\right)\left(\exp\left(c\left(\tau^{(q)}_l\right)^{-2}\right)\right)\Big/\left(\Omega^{(q)}_l\right)^2, \qquad \mu^{(q)}_l = \left(\nu^{(q)}_l\right)^{4/\Omega^{(q)}_l}, \qquad q = 2, 3,$$

if there exists a number $i \in I_l$ such that $k^{(q)}(i,l) \neq -1$, or

$$\nu_l^{(q)} = c\left(\Delta_1^2(l)+1\right), \qquad \mu_l^{(q)} = \nu_l^{(q)}, \qquad q=1,2,3,4,$$

otherwise. We also introduce the numbers

$$\alpha_{i,l}^{(q)} = \left(\eta_l^{(q)}+\mu_l^{(q)}\right)\exp(\exp(2i)), \qquad \beta_{i,l}^{(q)} = e^6\alpha_{i,l}^{(q)}, \qquad q=1,2,3,4, \qquad i\in I_l, \qquad l\in N,$$
$$\eta_l^{(q)} = \beta_{l\times 2^l,l}^{(q-1)} + 2^l, \qquad q=2,3,4, \qquad \eta_{l+1}^{(1)} = \beta_{l\times 2^l,l}^{(4)} + R_{l+1} + 2^{l+1}, \quad l\in N,$$
$$\gamma_{1,l}^{(3)} = e^2\alpha_{1,l}^{(3)}, \qquad \delta_{1,l}^{(3)} = e^4\alpha_{1,l}^{(3)}, \qquad \gamma_{1,l+1}^{(1)} = e^2\alpha_{1,l+1}^{(1)}, \qquad \delta_{1,l+1}^{(1)} = e^4\alpha_{1,l+1}^{(1)}, \quad l\in N.$$

Note that the numbers $\mu_l^{(q)}$, $l\in N$, $q=1,2,3,4$, have been chosen so as to provide the boundedness (proved in the second part of the present paper) of the coefficients of the Pfaff equation (1_1) to be constructed, and the numbers R_l, $l\in N$, have been chosen so as to provide the validity of the inequalities (10_1)–(13_2).

We introduce the "main" strips

$$\Pi^{(q)}(i,l) = \left\{t\in R^2_{>1}:\ \beta_{i,l}^{(q)} \le \zeta(t) \le \alpha_{i+1,l}^{(q)}\right\},$$
$$i\in\left\{1,\ldots,l\times 2^l-1\right\} \equiv I_l^1, \qquad l\in N, \qquad q=1,2,3,4,$$
$$\Pi^{(q)}\left(l\times 2^l,l\right) = \left\{t\in R^2_{>1}:\ \beta_{l\times 2^l,l}^{(q)} \le \zeta(t) \le \alpha_{1,l}^{(q+1)}\right\}, \qquad q=1,2,3, \qquad l\in N,$$
$$\Pi^{(4)}\left(l\times 2^l,l\right) = \left\{t\in R^2_{>1}:\ \beta_{l\times 2^l,l}^{(4)} \le \zeta(t) \le \alpha_{1,l+1}^{(1)}\right\}, \qquad l\in N,$$

the "transition" strips

$$P^{(q)}(i,l) = \left\{t\in R^2_{>1}:\ \alpha_{i,l}^{(q)} < \zeta(t) < \beta_{i,l}^{(q)}\right\}, \qquad i\in I_l, \qquad l\in N, \qquad q=1,2,3,4,$$

and the triangle $T = \left\{t\in R^2_{>1}:\ \zeta(t)\le \alpha_{1,1}^{(1)}\right\}$.

Each of the strips $P^{(3)}(1,l)$ and $P^{(1)}(1,l+1)$, $l\in N$, in which the transition is to be performed for the functions $\varphi(t)$ and $\psi(t)$ simultaneously, splits into the three substrips

$$P_1^{(3)}(1,l) = \left\{t\in R^2_{>1}:\ \alpha_{1,l}^{(3)} < \zeta(t) < \gamma_{1,l}^{(3)}\right\},$$
$$P_2^{(3)}(1,l) = \left\{t\in R^2_{>1}:\ \gamma_{1,l}^{(3)} \le \zeta(t) \le \delta_{1,l}^{(3)}\right\},$$
$$P_3^{(3)}(1,l) = \left\{t\in R^2_{>1}:\ \delta_{1,l}^{(3)} < \zeta(t) < \beta_{1,l}^{(3)}\right\}$$

and

$$P_1^{(1)}(1,l+1) = \left\{t\in R^2_{>1}:\ \alpha_{1,l+1}^{(1)} < \zeta(t) < \gamma_{1,l+1}^{(1)}\right\},$$
$$P_2^{(1)}(1,l+1) = \left\{t\in R^2_{>1}:\ \gamma_{1,l+1}^{(1)} \le \zeta(t) \le \delta_{1,l+1}^{(1)}\right\},$$
$$P_3^{(1)}(1,l+1) = \left\{t\in R^2_{>1}:\ \delta_{1,l+1}^{(1)} < \zeta(t) < \beta_{1,l+1}^{(1)}\right\},$$

respectively.

We also introduce the strips $\Pi L^{(q)}(i,l) \equiv P^{(q)}(i,l)\cup\Pi^{(q)}(i,l)\cup P^{(q)}(i+1,l)$, $i\in I_l^1$, $l\in N$, $q=1,2,3,4$, $\Pi L^{(q)}\left(l\times 2^l,l\right) \equiv P^{(q)}\left(l\times 2^l,l\right)\cup\Pi^{(q)}\left(l\times 2^l,l\right)\cup P^{(q+1)}(1,l)$, $l\in N$, $q=1,2,3$, and $\Pi L^{(4)}\left(l\times 2^l,l\right) \equiv P^{(4)}\left(l\times 2^l,l\right)\cup\Pi^{(4)}\left(l\times 2^l,l\right)\cup P^{(1)}(1,l+1)$, $l\in N$, and the sectors $S\Pi^{(q)}(i,l) = S^{(q)}(i,l)\cap\Pi L^{(q)}(i,l)$, $i\in I_l$, $l\in N$, $q=1,2,3,4$.

From (10_1) and (10_2), we obtain the estimates

$$\ln \psi_{i,l}^{(q)}(t) - (d, \ln t) \geq 0, \quad t \in \Pi L^{(q)}(i,l) \backslash S\Pi^{(q)}(i,l), \quad \forall d \in D(l), \quad i \in I_l, \quad l \in N, \quad q = 1,2, \quad (14_1)$$

$$\ln \psi_{i,l}^{(q)}(t) - (d, \ln t) \leq 0, \quad t \in \Pi L^{(q)}(i,l) \backslash S\Pi^{(q)}(i,l), \quad \forall d \in D(l), \quad i \in I_l, \quad l \in N, \quad q = 3,4, \quad (14_2)$$

respectively, by virtue of the inequalities $\|t\| \geq (t_1 + t_2)/\sqrt{2} \geq T_l$ valid in each strip $\Pi L^{(q)}(i,l)$.

In each strip $\Pi L^{(q)}(i,l)$, $i \in I_l$, $l \in N$, $q = 1,2,3,4$, we also have the inequalities

$$\sqrt{\|t\|} \geq \sqrt{(t_1 + t_2)/2} \geq \Delta_1(l), \quad t \in \Pi L^{(q)}(i,l), \quad i \in I_l, \quad l \in N, \quad q = 1,2,3,4. \qquad (15)$$

We introduce the notation $\Pi^{(q)}(l) = \Pi^{(q)}(1,l) \cup \left(\bigcup_{i=2}^{l \times 2^l} \left(P^{(q)}(i,l) \cup \Pi^{(q)}(i,l)\right)\right)$, $q = 1,2,3,4$, for the strips used in the realization of the left and right boundary lower exponent sets $\underline{D}_x(p') = D^{(1)}$ and $\underline{D}_x(p'') = D^{(2)}$ and the left and right boundary upper exponent sets $\bar{D}_x(\lambda') = D^{(3)}$ and $\bar{D}_x(\lambda'') = D^{(4)}$ as well as $\underline{\Pi}(l) = \Pi^{(1)}(l) \cup P^{(2)}(1,l) \cup \Pi^{(2)}(l)$ for the strips used in the realization of the lower characteristic set P_x and $\bar{\Pi}(l) = \Pi^{(3)}(l) \cup P^{(4)}(1,l) \cup \Pi^{(4)}(l)$ for the strips used in the realization of the characteristic set Λ_x.

Let us now proceed to the construction of the function $\varphi(t)$ used in the realization of the characteristic set Λ_x and the lower characteristic set P_x of the solution $x(t)$ to be constructed. We construct the function $\varphi(t)$ so as to ensure that its lower characteristic set P_φ coincides with the lower characteristic set $P \equiv \left\{p \in R_-^2 : \ p_1 + p_2 = -1\right\}$ of the function $\underline{E}(t) \equiv (e^{-t_1} + e^{-t_2})$ and its characteristic set Λ_φ coincides with the characteristic set $\Lambda \equiv \left\{\lambda \in R_+^2 : \ \lambda_1 + \lambda_2 = 3, \ 1 \leq \lambda_1 \leq 2\right\}$ of the function $\bar{E}(t) \equiv e^{\zeta(t)} \underline{E}^{-1}(t)$. In each strip $\underline{\Pi}(l)$, we define the function $\varphi(t)$ by the relation

$$\varphi(t) = \underline{E}(t), \qquad t \in \underline{\Pi}(l), \qquad l \in N, \qquad (16_1)$$

and in each strip $\bar{\Pi}(t)$, this function is given by the formula

$$\varphi(t) = \bar{E}(t), \qquad t \in \bar{\Pi}(l), \qquad l \in N. \qquad (16_2)$$

In the strip $P^{(3)}(1,l)$ lying between the strips $\underline{\Pi}(l)$ and $\bar{\Pi}(l)$, the function $\varphi(t)$ is given by the relations

$$\ln \varphi(t) = \ln \underline{E}(t) + (\zeta(t)/2 - \ln \underline{E}(t))\, e_{01}\left(\ln \zeta(t); \ln \alpha_{1,l}^{(3)}, \ln \gamma_{1,l}^{(3)}\right), \qquad t \in P_1^{(3)}(1,l),$$
$$\ln \varphi(t) = \zeta(t)/2 + \left(\ln \bar{E}(t) - \zeta(t)/2\right) e_{01}\left(\ln \zeta(t); \ln \delta_{1,l}^{(3)}, \ln \beta_{1,l}^{(3)}\right), \qquad t \in P_2^{(3)}(1,l) \cup P_3^{(3)}(1,l). \qquad (16_3)$$

In the strip $P^{(1)}(1,l+1)$ lying between the strips $\bar{\Pi}(l)$ and $\underline{\Pi}(l+1)$, we set

$$\ln \varphi(t) = \ln \bar{E}(t) + \left(\zeta(t)/2 - \ln \bar{E}(t)\right) e_{01}\left(\ln \zeta(t); \ln \alpha_{1,l+1}^{(1)}, \ln \gamma_{1,l+1}^{(1)}\right),$$
$$t \in P_1^{(1)}(1,l+1),$$
$$\ln \varphi(t) = \zeta(t)/2 + (\ln \underline{E}(t) - \zeta(t)/2)\, e_{01}\left(\ln \zeta(t); \ln \delta_{1,l+1}^{(1)}, \ln \beta_{1,l+1}^{(1)}\right),$$
$$t \in P_2^{(1)}(1,l+1) \cup P_3^{(1)}(1,l+1). \qquad (16_4)$$

Finally, in the triangle T and in the strip $P^{(1)}(1,1)$, the function $\varphi(t)$ is defined as follows:

$$\ln \varphi(t) = e_{01}\left(\ln \zeta(t); \ln \alpha_{1,1}^{(1)}, \ln \beta_{1,1}^{(1)}\right) \ln \underline{E}(t), \qquad t \in T \cup P^{(1)}(1,1). \qquad (16_5)$$

The function $\varphi(t)$ is thereby defined by formulas (16_1)–(16_5) in the entire quadrant $R_{>1}^2$.

Let us proceed to the construction of the function $\psi(t)$ used in the realization of the desired boundary exponent sets of the solution $x(t)$. First, in the triangle T, we set $\psi(t) = 1$. Now we define the function $\psi(t)$ in the strips $\Pi^{(q)}(l)$, $l \in N$, $q = 1, 2, 3, 4$. To this end, we introduce auxiliary functions $u_l^{(q)}(t)$ in the strips $\Pi^{(q)}(l)$, $l \in N$, $q = 1, 2, 3, 4$, by the relation

$$\begin{gathered} \ln u_l^{(q)}(t) = \ln \psi_{i,l}^{(q)}(t) + \left[\ln \psi_{i+1,l}^{(q)}(t) - \ln \psi_{i,l}^{(q)}(t)\right] e_{01}\left(\ln \zeta(t); \ln \alpha_{i+1,l}^{(q)}, \ln \beta_{i+1,l}^{(q)}\right), \\ t \in \Pi^{(q)}(i,l) \cup P^{(q)}(i+1,l) \cup \Pi^{(q)}(i+1,l), \qquad i \in I_l^1, \qquad l \in N, \qquad q = 1,2,3,4. \end{gathered} \tag{17}$$

In the case of the curve $D^{(q)}$ of the form (1_q) or (2_q), we define the desired function $\psi(t)$ in the strips $\Pi^{(q)}(l)$, $l \in N$, $q = 1, 2, 3, 4$, by the formula

$$\psi(t) = u_l^{(q)}(t), \qquad t \in \Pi^{(q)}(l), \qquad l \in N, \qquad q = 1,2,3,4, \tag{18$_1$}$$

and in the case of the curve $D^{(q)}$ of the form (3_q), we set

$$\begin{gathered} \ln \psi(t) = \ln u_l^{(q)}(t) + \left[\left(\Delta^{(q)}(0,0), \ln t\right) - \ln u_l^{(q)}(t)\right] \chi^{(q)}(t), \\ t \in \Pi^{(q)}(l), \qquad l \in N, \qquad q = 1,2,3,4, \end{gathered} \tag{18$_2$}$$

with the function

$$\begin{aligned} \chi^{(q)}(t) &= e_{01}\left((\ln t_2)/\left(\sqrt[3]{t_1}\ln t_1\right); 1, 3\right), \qquad q = 1,4, \\ \chi^{(q)}(t) &= e_{01}\left((\ln t_1)/\left(\sqrt[3]{t_2}\ln t_2\right); 1, 3\right), \qquad q = 2,3. \end{aligned}$$

To define the function $\psi(t)$ in the "transition" strips $P^{(q)}(1,l)$, $l \in N$, $q = 1, 2, 3, 4$, we first continue its values from the left and right sides of the strips $P^{(q)}(1,l)$ into these strips; more precisely, in the case of the curve $D^{(q)}$ of the form (3_q), we set

$$\begin{gathered} \ln \omega_{j,l}^{(q)}(t) = \ln \psi_{j,l}^{(q)}(t) + \left[\left(\Delta^{(q)}(0,0), \ln t\right) - \ln \psi_{j,l}^{(q)}(t)\right] \chi^{(q)}(t), \\ j = 1, \qquad t \in P^{(q)}(1,l), \qquad q = 1,2,3,4, \\ j = l \times 2^l, \qquad t \in P^{(q+1)}(1,l), \qquad q = 1,2,3, \\ t \in P^{(1)}(1,l+1), \qquad q = 4, \qquad l \in N, \end{gathered} \tag{19$_1$}$$

and in the case of the curve $D^{(q)}$ of the form (1_q) or (2_q), we set

$$\begin{aligned} \omega_{j,l}^{(q)}(t) = \psi_{j,l}^{(q)}(t), \qquad & j = 1, \qquad t \in P^{(q)}(1,l), \qquad q = 1,2,3,4, \\ & j = l \times 2^l, \qquad t \in P^{(q+1)}(1,l), \qquad q = 1,2,3, \\ & t \in P^{(1)}(1,l+1), \qquad q = 4, \qquad l \in N. \end{aligned} \tag{19$_2$}$$

In the strips $P^{(q)}(1,l)$, $l \in N$, $q = 2, 4$, that are not transition strips for the function $\varphi(t)$, we define the function $\psi(t)$ by the relation

$$\begin{gathered} \ln \psi(t) = \ln \omega_{l\times 2^l,l}^{(q-1)}(t) + \left[\ln \omega_{1,l}^{(q)}(t) - \ln \omega_{l\times 2^l,l}^{(q-1)}(t)\right] e_{01}\left(\ln \zeta(t); \ln \alpha_{1,1}^{(q)}, \ln \beta_{1,1}^{(q)}\right), \\ t \in P^{(q)}(1,l), \qquad q = 2,4. \end{gathered} \tag{18$_3$}$$

In the strip $P^{(3)}(1,l)$, $l \in N$, lying between the strips $\Pi^{(2)}(l)$ and $\Pi^{(3)}(l)$, we set

$$\ln \psi(t) = \ln \omega_{l\times 2^l,l}^{(2)}(t) + \left[\ln \omega_{1,l}^{(3)}(t) - \ln \omega_{l\times 2^l,l}^{(2)}(t)\right] e_{01}\left(\ln \zeta(t); \ln \gamma_{1,l}^{(3)}, \ln \delta_{1,l}^{(3)}\right), \quad t \in P^{(3)}(1,l). \tag{18$_4$}$$

In the strips $P^{(1)}(1,l+1)$, $l \in N$, lying between the strips $\Pi^{(4)}(l)$ and $\Pi^{(1)}(l+1)$, we define the function $\psi(t)$ by the formula

$$\begin{gathered} \ln \psi(t) = \ln \omega_{l\times 2^l,l}^{(4)}(t) + \left[\ln \omega_{1,l+1}^{(1)}(t) - \ln \omega_{l\times 2^l,l}^{(4)}(t)\right] e_{01}\left(\ln \zeta(t); \ln \gamma_{1,l+1}^{(1)}, \ln \delta_{1,l+1}^{(1)}\right), \\ t \in P^{(1)}(1,l+1). \end{gathered} \tag{18$_5$}$$

Finally, in the strip $P^{(1)}(1,1)$, we set

$$\ln\psi(t) = e_{01}\left(\ln\zeta(t); \ln\alpha_{1,1}^{(1)}, \ln\beta_{1,1}^{(1)}\right)\ln\omega_{1,1}^{(1)}(t), \qquad t\in P^{(1)}(1,1). \tag{18_6}$$

The function $\psi(t)$ is thereby defined by formulas (18_1)–(18_6) in the entire quadrant $R^2_{>1}$.

3. THE CONSTRUCTION OF THE CHARACTERISTIC AND LOWER CHARACTERISTIC SETS OF THE SOLUTION $x(t)$

3.1. The Construction of the Lower Characteristic Set of the Function $\varphi(t)$

Let us show that the lower characteristic set P_φ of the function $\varphi(t)$ coincides with P. We choose an arbitrary vector $p\in P$ and set $R_\varphi(p,d,t)\equiv \ln\varphi(t)-(p,t)-(d,\ln t)$. Let us prove the relation $\underline{l}_\varphi(p)=\underline{\lim}_{t\to\infty} R_\varphi\left((p_1,-1-p_1),0,t\right)/\|t\|=0$, which implies the first condition in definition (3) of the lower characteristic vector. Let the lower limit $\underline{l}_\varphi(p)$ be realized along a sequence $\{t(m)\}\uparrow+\infty$, and let $m\in N$ be given. Without loss of generality, one can restrict considerations to the following four cases:

(1) $t(m)\in\underline{\Pi}\left(l_m\right)$;
(2) $t(m)\in\bar{\Pi}\left(l_m\right)$;
(3) $t(m)\in P^{(3)}\left(1,l_m\right)$;
(4) $t(m)\in P^{(1)}\left(1,l_m+1\right)$.

If the first case takes place, then from (16_1) with $t_1(m)\ge t_2(m)$, we obtain the estimates

$$\begin{aligned}R_\varphi\left((p_1,-1-p_1),0,t(m)\right) &= \ln\left(e^{-t_1(m)}+e^{-t_2(m)}\right)-p_1t_1(m)+(1+p_1)\,t_2(m)\\ &\ge -t_2(m)-p_1t_1(m)+(1+p_1)\,t_2(m)=p_1\left(t_2(m)-t_1(m)\right)\ge 0,\end{aligned}$$

and, for $t_1(m)<t_2(m)$, we obtain the estimates

$$\begin{aligned}R_\varphi\left((p_1,-1-p_1),0,t(m)\right) &\ge -t_1(m)-p_1t_1(m)+(1+p_1)\,t_2(m)\\ &=(1+p_1)(t_2(m)-t_1(m))\ge 0.\end{aligned}$$

In the second case, relation (16_2) implies that

$$\begin{aligned}&R_\varphi\left((p_1,-1-p_1),0,t(m)\right)\\ &\quad = t_1(m)+t_2(m)-\ln\left(e^{-t_1(m)}+e^{-t_2(m)}\right)-p_1t_1(m)+(1+p_1)\,t_2(m)\ge 0.\end{aligned}$$

In cases (3) and (4), in a similar way, by using relations (16_3) and (16_4), one can obtain the estimates $R_\varphi\left((p_1,-1-p_1),0,t(m)\right)\ge 0$. This completes the proof of the necessary inequality $\underline{l}_\varphi(p)\ge 0$. The relation $\underline{l}_\varphi(p)=0$ and the second condition in definition (3) of the lower characteristic vector are realized along a sequence $\{t(l)\}_{l\in N}$ such that $t(l)\in\underline{\Pi}(l)$, $t_1(l)=t_2(l)$, $l\in N$. We have thereby proved the inclusion $P\subset P_\varphi$. It remains to prove the opposite inclusion $P_\varphi\subset P$. We choose an arbitrary lower characteristic vector $p\in P_\varphi$. Then, from the first condition in definition (3), along the above-constructed sequence $\{t(l)\}_{l\in N}$, we obtain the inequality $p_1+p_2\le -1$. Likewise, from the first condition in definition (3), along the sequences $\{t'(l)\}_{l\in N}$, $t'(l)\in\underline{\Pi}(l)$, $t_2'(l)=e$, $t_1'(l)\uparrow+\infty$ as $l\to\infty$, and $\{t''(l)\}_{l\in N}$, $t''(l)\in\underline{\Pi}(l)$, $t_1''(l)=e$, $t_2''(l)\uparrow+\infty$ as $l\to\infty$, we obtain the inequalities $p_1\le 0$ and $p_2\le 0$, respectively. Since the lower characteristic set P_φ can be represented [4] by a strictly monotone decreasing curve, we find that it coincides with P.

3.2. The Construction of the Characteristic Set of the Function $\varphi(t)$

Let us show that the characteristic set Λ_φ of the function $\varphi(t)$ coincides with Λ. We first prove that an arbitrary vector $\lambda\in\Lambda$ is the characteristic vector of the function $\varphi(t)$. By the construction (16_5) of the function $\varphi(t)$, in the triangle T, we have the obvious estimate

$$R_\varphi\left((\lambda_1,3-\lambda_1),0,t\right)\le 0.$$

The inequalities $R_\varphi\left((\lambda_1, 3-\lambda_1), 0, t\right) \le 0$ are valid in the strip $P^{(1)}(1,1)$ as well as in the strips $\underline{\Pi}(l)$, $l \in N$, by virtue of (16_5) and (16_1). The representation (16_2) of the function $\varphi(t)$ in the strips $\bar{\Pi}(l)$, $l \in N$, implies the estimates

$$\begin{aligned} R_\varphi\left((\lambda_1, 3-\lambda_1), 0, t\right) &= t_1 + t_2 - \ln\left(e^{-t_1} + e^{-t_2}\right) - \lambda_1 t_1 + (\lambda_1 - 3) t_2 \\ &\le (2-\lambda_1) t_1 + (\lambda_1 - 2) t_2 = (2-\lambda_1)(t_1 - t_2) \le 0, \qquad t_1 \le t_2, \\ R_\varphi\left((\lambda_1, 3-\lambda_1), 0, t\right) &\le (1-\lambda_1) t_1 + (\lambda_1 - 1) t_2 = (\lambda_1 - 1)(t_2 - t_1) \le 0, \qquad t_1 > t_2. \end{aligned}$$

Finally, from the representations (16_3) and (16_4), we obtain the last needed estimate

$$R_\varphi\left((\lambda_1, 3-\lambda_1), 0, t\right) \le 0, \qquad t \in P^{(3)}(1,l) \cup P^{(1)}(1,l+1), \qquad l \in N.$$

We have thereby proved the inequality $\bar{l}_\varphi(\lambda) \le 0$. Moreover, the relation $\bar{l}_\varphi(\lambda) = 0$ and, obviously, the second inequality in definition (2) of the characteristic vector are valid in the direction $t \in \bar{\Pi}(l)$, $t_2 = t_1$, $l \in N$. Consequently, $\Lambda \subset \Lambda_\varphi$. Let us now prove the opposite inclusion $\Lambda_\varphi \subset \Lambda$. Let $\lambda \in \Lambda_\varphi$. Then, from the first condition in definition (2) of the characteristic vector, we obtain the inequality $3 - \lambda_1 - \lambda_2 \le 0$ in the direction $t \in \bar{\Pi}(l)$, $t_1 = t_2$, $l \in N$, and the inequalities $\lambda_1 \ge 1$ and $\lambda_2 \ge 1$ in the directions $t_2 = e$, $t_1 \to \infty$, $t \in \bar{\Pi}(l)$, $l \in N$, and $t_1 = e$, $t_2 \to \infty$, $t \in \bar{\Pi}(l)$, $l \in N$, respectively. Since the set Λ_φ can be represented [3] by a strictly monotone decreasing curve, we find that it necessarily coincides with Λ.

3.3. Proof of the Coincidence of the Lower Characteristic Set and the Characteristic Set of the Solution $x(t)$ with the Sets P and Λ, Respectively

Let us show that the function $\psi(t)$ does not affect the characteristic set and the lower characteristic set of the solution $x(t)$; more precisely, $P_x = P_\varphi$ and $\Lambda_x = \Lambda_\varphi$. To this end, we prove the existence of the limit

$$\lim_{t\to\infty} \left(\|t\|^{-1} \ln \psi(t)\right) = 0. \tag{20}$$

We take an arbitrary $q = 1,2,3,4$. Then, by (17), ($8_{1,1}$)–($8_{2,2}$), and (15), we have the estimates

$$\begin{aligned} \left|\ln u_l^{(q)}(t)\right| &= \left|\ln \psi_{i,l}^{(q)}(t)\right| \le \left\|\Delta^{(q)}(i,l)\right\| \|\ln t\| + \|\ln t\|^2 \\ &\le \sqrt{\|t\|}\, \|\ln t\| + \|\ln t\|^2, \qquad t \in \Pi^{(q)}(i,l), \end{aligned} \tag{21_1}$$

in the "main" strips $\Pi^{(q)}(i,l)$, $i \in I_l$, $l \in N$. Likewise, in the "transition" strips $P^{(q)}(i+1,l)$, $i \in I_l^1$, $l \in N$, from (17) and (15), we obtain

$$\left|\ln u_l^{(q)}(t)\right| \le 2\left|\ln \psi_{i,l}^{(q)}(t)\right| + \left|\ln \psi_{i+1,l}^{(q)}(t)\right| \le 3\left(\sqrt{\|t\|}\, \|\ln t\| + \|\ln t\|^2\right), \qquad t \in P^{(q)}(i+1,l). \tag{21_2}$$

Therefore, in the strips $\Pi^{(q)}(l)$ from (18_1), (21_1), and (21_2) in the case of the curve $D^{(q)}$ of the form (1_q) or (2_q), we have the estimates

$$|\ln \psi(t)| \le 3\left(\sqrt{\|t\|}\, \|\ln t\| + \|\ln t\|^2\right), \qquad t \in \Pi^{(q)}(l), \qquad l \in N, \qquad q = 1,2,3,4. \tag{22_1}$$

But if the curve $D^{(q)}$ has the form (3_q), then, by virtue of the inequality $\left\|\Delta^{(q)}(0,0)\right\| \le \sqrt{\|t\|}$ for sufficiently large values of the norm $\|t\| \ge \zeta_1$, and (18_2), we have the estimates

$$\begin{aligned} |\ln \psi(t)| &\le 6\left(\sqrt{\|t\|}\, \|\ln t\| + \|\ln t\|^2\right) + \left\|\Delta^{(q)}(0,0)\right\| \|\ln t\| \\ &\le 7\left(\sqrt{\|t\|}\, \|\ln t\| + \|\ln t\|^2\right), \qquad t \in \Pi^{(q)}(l), \qquad \|t\| \ge \zeta_1, \qquad q = 1,2,3,4. \end{aligned} \tag{22_2}$$

Finally, in the "transition" strips $P^{(q)}(1,l)$, $l \in N$, $q = 1,2,3,4$, on the basis of relations (18_3)–(18_6), (19_1), and (19_2), we obtain the estimate

$$|\ln \psi(t)| \le 9\left(\sqrt{\|t\|}\, \|\ln t\| + \|\ln t\|^2\right), \qquad t \in P^{(q)}(1,l), \qquad \|t\| \ge \zeta_1, \qquad q = 1,2,3,4. \tag{22_3}$$

Inequalities (22_1)–(22_3) imply the estimate

$$|\ln\psi(t)| \le 9\left(\sqrt{\|t\|}\,\|\ln t\| + \|\ln t\|^2\right) \le \|t\|^{2/3}, \qquad t \in R^2_{>1}, \qquad \|t\| \ge \zeta_0 \ge \zeta_1, \tag{23}$$

which implies (20). We have thereby shown that the lower characteristic set P_x and the characteristic set Λ_x of the solution $x(t)$ of Eq. (1_1) coincide with P and Λ, respectively.

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