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A Joint Description of the Boundary Exponent Sets of a Solution of a Linear Pfaff System: II

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In the present paper, we complete the proof of Theorem 1 stated in the first part [1] and below. We continue the numbering of formulas in [1].

Theorem 1. *Let the following objects be given*:

an arbitrary positive integer n;

arbitrary closed concave monotone decreasing curves $D^{(1)}$ *and* $D^{(2)}$ *on the two-dimensional plane unbounded on the right and below and on the left and above, respectively, and having slopes not less* (*respectively, greater*) *than* -1 *at each interior point*;

arbitrary closed convex monotone decreasing curves $D^{(3)}$ *and* $D^{(4)}$ *on the two-dimensional plane unbounded on the right and below and on the left and above, respectively, and having slopes not greater* (*respectively, less*) *than* -1 *at each interior point.*

Then there exists a completely integrable Pfaff system (1_n) *with infinitely differentiable bounded coefficients such that its arbitrary nontrivial solution* $x : R^2_{>1} \to R^n \backslash \{0\}$ *has the left and right boundary lower exponent sets* $\underline{D}_x(p') = D^{(1)}$ *and* $\underline{D}_x(p'') = D^{(2)}$, *respectively, and the left and right boundary upper exponent sets* $\bar{D}_x(\lambda') = D^{(3)}$ *and* $\bar{D}_x(\lambda'') = D^{(4)}$, *respectively.*

Proof of Theorem 1 (continued)**.**

4. CONSTRUCTION OF THE BOUNDARY EXPONENT SETS

In this section, we construct the left and right boundary lower and upper exponent sets of the nontrivial solution of the Pfaff equation (1_1) constructed in Section 2 of [1].

4.1. Proof of the Coincidence of the Left and Right Boundary Lower Exponent Sets $\underline{D}_x(p')$ *and* $\underline{D}_x(p'')$ *with the Curves* $D^{(1)}$ *and* $D^{(2)}$*, Respectively*

We take an arbitrary point $d = (d_1, d_2)$ of the everywhere dense partition $D^{(1)}_\infty$ of the curve $D^{(1)}$ and show that it belongs to the left boundary lower exponent set $\underline{D}_x(p')$.

Following [2], we introduce the notation

$$\beta^{(1)}(d) \equiv \varliminf_{t\to\infty} \frac{\ln|x(t)| + t_1 - (d, \ln t)}{\|\ln t\|}$$

for the lower limit occurring in definition (5), where $p' = (-1, 0)$ is the left boundary point of the lower characteristic set P_x.

Let us prove the inequality $\beta^{(1)}(d) \le 0$. By the definition of the partition $D^{(1)}_\infty$, there exists an index $l(d) \in N$ such that $d \in D^{(1)}_l$ for all $l \ge l(d)$ and $d \notin D^{(1)}_l$ for all $l < l(d)$. Suppose that d is the i_mth point of the $(l(d)+m)$th partition, $m \in N$. In each strip $\Pi^{(1)}(i_m, l(d)+m)$, $m \in N$, in which the function $\ln\psi(t)$ is defined on the basis of the point $d = \Delta^{(1)}(i_m, l(d)+m)$, we take a point $\tau(m)$ on the curve $\ln t_2/\ln t_1 = 1/\left|k^{(1)}(d)\right| > 1$ if the slope $k^{(1)}(d)$ of the curve $D^{(1)}$ at the point d is not equal to -1 and on the line $t_2/t_1 = 2e$ otherwise. We thereby obtain a sequence $\{\tau(m)\} \uparrow +\infty$ such that

$$\ln \psi(\tau(m)) = (d, \ln \tau(m)), \qquad \ln \varphi(\tau(m)) = \ln \underline{E}(\tau(m)),$$
$$\lim_{m\to\infty} \frac{\ln x(\tau(m)) + \tau_1(m) - (d, \ln \tau(m))}{\|\ln \tau(m)\|} = 0,$$

which implies the desired inequality $\beta^{(1)}(d) \le 0$. Note that, in the case of the curve $D^{(1)}$ of the form (3_1), the validity of the relation $\ln \psi(\tau(m)) = (d, \ln \tau(m))$ is provided by (13_1).

Let the lower limit $\beta^{(1)}(d)$ be realized along a sequence $\{t(m)\} \uparrow +\infty$ such that $t_j(m) \to +\infty$ as $m \to +\infty$, $j = 1, 2$. The existence of such a sequence follows from Lemma 1 in [3]. Without loss of generality, we can assume that

$$|(d, \ln t(m))| \le \|t(m)\|^{2/3}, \qquad m \in N, \tag{24_1}$$
$$t_1(m) \ge \max\{\zeta_0, 128\}, \qquad m \in N, \tag{24_2}$$

and also, in the case of the curve $D^{(1)}$ of the form (3_1),

$$\left(\Delta^{(1)}(0,0) - d, \left(1, \sqrt[3]{t_1(m)}\right)\right) \ge 0, \qquad m \in N. \tag{24_3}$$

Without loss of generality, we can assume that all elements $t(m)$ of this sequence belong to strips of the quadrant $R^2_{>1}$ with distinct indices l_m, $1 < l_m < l_{m+1} \to +\infty$ as $m \to +\infty$, and

$$d \in D^{(1)}(l_m), \qquad m \in N. \tag{25}$$

Let us now prove the inequality $\beta^{(1)}(d) \ge 0$. If the sequence $\{t(m)\}$ contains an infinite subsequence $\{t(m_j)\}$ such that each point $t(m_j)$ satisfies the estimate $\ln \psi(t(m_j)) - (d, \ln t(m_j)) \ge 0$, then, by virtue of the inequality $\ln \varphi(t(m)) + t_1(m) \ge 0$, we have $\beta(d) \ge 0$. Therefore, without loss of generality, just as in [2], we can assume that

$$\ln \psi(t(m)) - (d, \ln t(m)) < 0 \qquad \forall m \in N. \tag{26}$$

We take some $m \in N$ and suppose that $t(m)$ lies in the strip $\Pi^{(1)}(l_m)$ used for the realization of the left boundary lower exponent set $\underline{D}_x(p') = D^{(1)}$. Note that the solution $x(t)$ has been constructed so as to preserve the idea of proof of the inequality $\beta^{(1)}(d) \ge 0$ in [2] in the realization of the left boundary lower exponent set $\underline{D}_x(p') = D^{(1)}$ in the strips $\Pi^{(1)}(l_m)$, which are the main strips of the set $\underline{D}_x(p')$. Since in the case of the slope equal to -1 of the curve $D^{(1)}$ at some point $\Delta^{(1)}(i,l) \in D^{(1)}_l$, the function $\psi^{(1)}_{i,l}(t)$ is defined in another way than in [2], we still give a concise proof of the inequality $\beta^{(1)}(d) \ge 0$ in the strips $\Pi^{(1)}(l_m)$. This proof substantially uses the fact that the validity of the estimate (12_1) is achieved in the strips corresponding to the points $\Delta^{(1)}(i,l)$ at which the slope of $D^{(1)}$ is equal to -1.

Let the curve $D^{(1)}$ have the form (1_1) or (2_1). If $t(m)$ belongs to the "main" strip $\Pi^{(1)}(i_m, l_m)$, then it follows from (26), (18_1), and (17) that $\ln \psi^{(1)}_{i_m,l_m}(t(m)) - (d, \ln t(m)) < 0$, which, together with (14_1) and (25), implies that $t(m) \in S\Pi^{(1)}(i_m, l_m)$. Since $D^{(1)}$ is concave, we find that the point $d \in D^{(1)}$ does not lie above the tangent to the curve $D^{(1)}$ at the point $\Delta^{(1)}(i_m, l_m) \in D^{(1)}$, i.e.,

$$\Delta^{(1)}_1(i_m, l_m) - d_1 + \Theta^{(1)}_{i_m,l_m}\left(\Delta^{(1)}_2(i_m, l_m) - d_2\right) \ge 0. \tag{27}$$

Let us estimate $R_x((-1,0),d,t(m))$ from below. The inclusion $t(m) \in S\Pi^{(1)}(i_m,l_m)$, together with relations $(8_{1,1})$ and $(8_{1,2})$ and inequalities (12_1) and (27), implies the estimates

$$\begin{aligned} &R_x((-1,0),d,t(m)) \\ &\quad= \ln \underline{E}(t(m)) + t_1(m) + \ln \psi^{(1)}_{i_m,l_m}(t(m)) - (d, \ln t(m)) \geq \left(\Delta^{(1)}(i_m,l_m) - d, \ln t(m)\right) \\ &\quad= \Bigg[\left\{\left(\Delta^{(1)}_1(i_m,l_m) - d_1\right) + \Theta^{(1)}_{i_m,l_m}\left(\Delta^{(1)}_2(i_m,l_m) - d_2\right)\right\} \\ &\qquad + \left(\Delta^{(1)}_2(i_m,l_m) - d_2\right)\left(\frac{\ln t_2(m)}{\ln t_1(m)} - \Theta^{(1)}_{i_m,l_m}\right)\Bigg] \ln t_1(m) \\ &\quad\geq -\left|\Delta^{(1)}_2(i_m,l_m) - d_2\right| \left|\frac{\ln t_2(m)}{\ln t_1(m)} - \Theta^{(1)}_{i_m,l_m}\right| \ln t_1(m) \geq -2^{-l_m}\|\ln t(m)\|. \end{aligned}$$

Let the point $t(m)$ lie in the "transition" strip $P^{(1)}(i_m+1,l_m)$, $i_m \in I^1_{l_m}$. If the relation

$$\psi^{(1)}_{i_m+1,l_m}(t(m)) \geq \psi^{(1)}_{i_m,l_m}(t(m))$$

is valid at $t(m)$, then from (26), (18_1), and (17), we obtain $\ln \psi^{(1)}_{i_m,l_m}(t(m)) - (d, \ln t(m)) < 0$. Just as for the case in which $t(m)$ lies in the "main" strip $\Pi^{(1)}(i_m,l_m)$, from this inequality, one can obtain the estimate

$$R_x((-1,0),d,t(m)) \geq -2^{-l_m}\|\ln t(m)\|. \tag{28}$$

But if $\psi^{(1)}_{i_m+1,l_m}(t(m)) < \psi^{(1)}_{i_m,l_m}(t(m))$ at $t(m)$, then $\ln \psi^{(1)}_{i_m+1,l_m}(t(m)) - (d, \ln t(m)) < 0$. By performing considerations similar to the case of the inclusion $t(m) \in \Pi^{(1)}(i_m,l_m)$ and by writing out the equation of the tangent to the curve $D^{(1)}$ at the point $\Delta^{(1)}(i_m+1,l_m)$ rather than $\Delta^{(1)}(i_m,l_m)$, we obtain (28).

Now let the curve $D^{(1)}$ have the form (3_1), and let $t(m) \in \Pi^{(1)}(l_m)$. If either

$$\ln t_2(m) \leq \sqrt[3]{t_1(m)} \ln t_1(m)$$

or

$$\ln t_2(m) > \sqrt[3]{t_1(m)} \ln t_1(m) \quad \text{and} \quad \ln u^{(1)}_{l_m}(t(m)) \leq \left(\Delta^{(1)}(0,0), \ln t(m)\right),$$

then, by virtue of the relation $\psi(t(m)) \geq u^{(1)}_{l_m}(t(m))$, we obtain the estimate (28). But if $\ln t_2(m) > \sqrt[3]{t_1(m)} \ln t_1(m)$ and $\ln u^{(1)}_{l_m}(t(m)) > \left(\Delta^{(1)}(0,0), \ln t(m)\right)$, then it follows from (24_3) that

$$\begin{aligned} R_x((-1,0),d,t(m)) &\geq \left(\Delta^{(1)}(0,0) - d, \ln t(m)\right) \\ &\geq \left(\Delta^{(1)}(0,0) - d, \left(1, \sqrt[3]{t_1(m)}\right)\right) \ln t_1(m) \geq 0. \end{aligned}$$

We have thereby proved inequality (28) for $R_x((-1,0),d,t(m))$ for the case in which $t(m)$ belongs to $\Pi^{(1)}(l_m)$.

Now we suppose that $t(m)$ belongs to the strip $\Pi^{(2)}(l_m)$ used for the realization of the right boundary lower exponent set $\underline{D}_x(p'') = D^{(2)}$. Let $D^{(2)}$ have the form (1_2) or (2_2). If $t(m) \in \Pi^{(2)}(i_m,l_m)$, then, by definitions (18_1) and (17) of the function $\psi(t)$, inequality (26) in this strip acquires the form $\ln \psi^{(2)}_{i_m,l_m}(t(m)) - (d, \ln t(m)) < 0$. Since $d \in D(l)$, it follows from (14_1) that $t(m) \in S\Pi^{(2)}(i_m,l_m)$. This inclusion, together with (11_2), implies that $t_2(m) \leq t_1(m)/e$. Then, by (23), (24_1), and (24_2), we have the estimates

$$\begin{aligned} R_x((-1,0),d,t(m)) &\geq \ln \underline{E}(t(m)) + t_1(m) + \ln \psi(t(m)) - (d, \ln t(m)) \\ &\geq -t_2(m) + t_1(m) - 2\|t(m)\|^{2/3} \geq t_1^{2/3}(m)\left(t_1^{1/3}(m) - 4\sqrt[3]{2}\right)\Big/2 \geq 0. \end{aligned} \tag{29}$$

If $t(m) \in P^{(2)}(i_m+1, l_m)$, $i \in I^1_{l_m}$, and $\psi^{(2)}_{i_m+1,l_m}(t(m)) \ge \psi^{(2)}_{i_m,l_m}(t(m))$, then it follows from (18_1), (17), and (26) that $\ln \psi^{(2)}_{i_m,l_m}(t(m)) - (d, \ln t(m)) < 0$. In this case, we obtain the estimate (29) just as for the case in which $t(m)$ belongs to the strip $\Pi^{(2)}(i_m, l_m)$. But if $t(m)$ lies in the strip $P^{(2)}(i_m+1, l_m)$ and $\psi^{(2)}_{i_m+1,l_m}(t(m)) < \psi^{(2)}_{i_m,l_m}(t(m))$, then it follows from (26) and (14_1) that $t(m)$ belongs to $S\Pi^{(2)}(i_m+1, l_m)$, which, together with (11_2), implies that $t_2(m) \le t_1(m)/e$. Then the estimate (29) is also satisfied. If $D^{(2)}$ is a curve of the form (3_2) and $\ln t_1(m) \le \sqrt[3]{t_2(m)} \ln t_2(m)$, then the estimate (29) can be proved by virtue of the relation $\psi(t(m)) = u^{(2)}_{l_m}(t(m))$. Now let $\ln t_1(m) > \sqrt[3]{t_2(m)} \ln t_2(m)$. Then, by (13_2), $t_1(m) > t_2(m)^{\sqrt[3]{t_2(m)}} \ge 2et_2(m)$, which allows one to obtain the estimate (29) in a similar way.

This completes the proof of the estimate (29) for $R_x((-1,0), d, t(m))$ for the case in which $t(m) \in \Pi^{(2)}(l_m)$.

Now let $t(m)$ lie in the strips $\bar{\Pi}(l_m)$ used in the realization of upper boundary exponent sets. By estimating $R_x((-1,0), d, t(m))$ from below, from (23), (24_1), and (24_2), we obtain the inequalities

$$\begin{aligned} R_x((-1,0), d, t(m)) &\ge \ln \bar{E}(t(m)) + \ln \psi(t(m)) + t_1(m) - (d, \ln t(m)) \\ &\ge 2t_1(m) + t_2(m) - \ln 2 - 2\|t(m)\|^{2/3} \ge t_1(m) - \ln 2 + \|t(m)\| - 2\|t(m)\|^{2/3} \ge 0. \end{aligned} \tag{30}$$

If $t(m)$ belongs to the strip $P^{(2)}(1, l_m)$ lying between the strips $\Pi^{(1)}(l_m)$ and $\Pi^{(2)}(l_m)$, then, by (18_3), either $\psi(t(m)) \ge \omega^{(1)}_{l_m \times 2^{l_m}, l_m}(t(m))$ or $\psi(t(m)) \ge \omega^{(2)}_{1,l_m}(t(m))$. If the first inequality is valid, then, by using considerations similar to those in the case of the inclusion $t(m) \in \Pi^{(1)}(l_m)$, one can prove the estimate (28). In the case of the second inequality, by using the relation $\varphi(t(m)) = \underline{E}(t(m))$ and by following the line of argument for the case in which $t(m) \in \Pi^{(2)}(l_m)$, one can obtain the estimate (29).

Consider the case in which $t(m)$ belongs to the strip $P^{(3)}(1, l_m)$ lying between the strips $\underline{\Pi}(l_m)$ and $\bar{\Pi}(l_m)$. First, let $t(m) \in P^{(3)}_1(1, l_m)$. Then, by (18_4) and (16_3), $\psi(t(m)) = \omega^{(2)}_{l_m \times 2^{l_m}, l_m}(t(m))$, and either $\varphi(t(m)) \ge \underline{E}(t(m))$ or $\ln \varphi(t(m)) \ge \zeta(t(m))/2$. If the first inequality is valid, then, just as in the case of the inclusion $t(m) \in \Pi^{(2)}(l_m)$, we obtain the estimate (29). In the case of the second inequality, by using (24_1), (24_2), and (23), we obtain the estimates

$$\begin{aligned} R_x((-1,0), d, t(m)) &\ge \zeta(t(m))/2 + t_1(m) + \ln \psi(t(m)) - (d, \ln t(m)) \\ &\ge (3t_1(m) + t_2(m))/2 - 2\|t(m)\|^{2/3} \ge \|t(m)\|/2 - 2\|t(m)\|^{2/3} \ge 0. \end{aligned} \tag{31}$$

If $t(m)$ lies in the strip $P^{(3)}_2(1, l_m)$, then $\ln \varphi(t(m)) = \zeta(t(m))/2$; consequently, the estimate (31) is valid. But if $t(m) \in P^{(3)}_3(1, l_m)$, then $\psi(t(m)) = \omega^{(3)}_{1,l_m}(t(m))$ and either $\ln \varphi(t(m)) \ge \zeta(t(m))/2$ or $\varphi(t(m)) \ge \bar{E}(t(m))$. In the case of the first inequality, we arrive at the estimate (31). In the case of the second inequality, by an argument similar to that used for the inclusion $t(m) \in \bar{\Pi}(l_m)$, one can prove the estimate (30).

Finally, we consider the case in which $t(m)$ belongs to the strip $P^{(1)}(1, l_m+1)$ lying between the strips $\bar{\Pi}(l_m)$ and $\underline{\Pi}(l_m+1)$. If $t(m) \in P^{(1)}_1(1, l_m+1)$, then from (18_5), we obtain the relation $\psi(t(m)) = \omega^{(4)}_{l_m \times 2^{l_m}, l_m}(t(m))$, and relation (16_4) implies that at least one of the inequalities $\varphi(t(m)) \ge \bar{E}(t(m))$ and $\ln \varphi(t(m)) \ge \zeta(t(m))/2$ is valid. Then the estimate (30) or (31), respectively, holds. If $t(m) \in P^{(1)}_2(1, l_m+1)$, then $\ln \varphi(t(m)) = \zeta(t(m))/2$; consequently, the estimate (31) is valid. If $t(m)$ lies in the strip $P^{(1)}_3(1, l_m+1)$, then $\psi(t(m)) = \omega^{(1)}_{1,l_m+1}(t(m))$, and either $\ln \varphi(t(m)) \ge \zeta(t(m))/2$ or $\varphi(t(m)) \ge \underline{E}(t(m))$. We obtain the estimate (31) in the first case and the estimate (28) in the other.

We have thereby proved the estimate (28) and the necessary property $\beta^{(1)}(d) \ge 0$ for all $m \in N$, $1 < l_m \to +\infty$ as $m \to \infty$. The second condition in definition (5) of the lower characteristic exponent of the vector d is realized on the above-constructed sequence $\tau(m)$. We have thereby proved the inclusion $D_\infty^{(1)} \subset \underline{D}_x(p')$, which implies that the set $D_\infty^{(1)}$ is everywhere dense on the curve $D^{(1)}$, the curves $D^{(1)}$ and $\underline{D}_x(p')$ are continuous, and $D^{(1)}$ is contained in the set $\underline{D}_x(p')$. In the case of a curve $D^{(1)}$ of the form (1_1) or (2_1), the last fact implies that $\underline{D}_x(p') = D^{(1)}$.

Now let us show that, in the case of a curve $D^{(1)}$ of the form (3_1), the left boundary lower exponent set $\underline{D}_x(p')$ is not larger than this curve. We take an arbitrary exponent $d \in \underline{D}_x(p')$. Then we obtain the inequality $\Delta_2^{(1)}(0,0) - d_2 \geq 0$ along some sequence $\{t(m)\} \uparrow +\infty$ of the form $t(m) = \left(t_1(m), t_1(m)^3 \sqrt[3]{t_1(m)}\right)$, $t(m) \in \Pi^{(1)}(l_m)$. Since $\underline{D}_x(p')$ is a closed strictly monotone decreasing curve, we have the coincidence $\underline{D}_x(p') = D^{(1)}$ in this case as well.

By the above-represented method, one can also prove the coincidence of the right boundary lower exponent set $\underline{D}_x(p'')$ with the given curve $D^{(2)}$.

4.2. The Proof of the Coincidence of the Left and Right Boundary Upper Exponent Sets $\bar{D}_x(\lambda')$ and $\bar{D}_x(\lambda'')$ with the Curves $D^{(3)}$ and $D^{(4)}$, Respectively

For example, let us show that $\bar{D}_x(\lambda') = D^{(3)}$. The relation $\bar{D}_x(\lambda'') = D^{(4)}$ can be proved in a similar way. We take an arbitrary point d of the partition $D_\infty^{(3)}$ of the curve $D^{(3)}$ and show that it belongs to the set $\bar{D}_x(\lambda')$.

We set

$$\beta^{(3)}(d) \equiv \overline{\lim_{t\to\infty}} \frac{\ln|x(t)| - t_1 - 2t_2 - (d, \ln t)}{\|\ln t\|}$$

for the upper limit occurring in definition (4) of the left upper characteristic exponent and show that $\beta^{(3)}(d) = 0$.

It follows from the definition of the partition $D_\infty^{(3)}$ that there exists an index $l(d) \in N$ such that $d \in D_l^{(3)}$ for all $l \geq l(d)$ and $d \notin D_l^{(3)}$ for all $l < l(d)$. Let d be the i_mth point of the $(l(d)+m)$th partition, $m \in N$. In each strip $\Pi^{(3)}(i_m, l(d)+m)$ in which the function $\ln\psi(t)$ is defined on the basis of the point $d = \Delta^{(3)}(i_m, l(d)+m)$, we take a point $\tau(m)$ defined as follows. If the slope $k^{(3)}(d)$ of the curve $D^{(3)}$ at the point d is not equal to -1, then $\tau(m)$ is chosen on the curve $\ln t_2/\ln t_1 = 1/\left|k^{(3)}(d)\right| < 1$; otherwise, we choose it on the line $t_2/t_1 = 1/(2e)$. We obtain a sequence $\{\tau(m)\} \uparrow +\infty$ such that, by (13_2), $\ln\psi(\tau(m)) = (d, \ln\tau(m))$, $\ln\varphi(\tau(m)) = \bar{E}(\tau(m))$, and

$$\lim_{m\to\infty} \frac{\ln x(\tau(m)) - \tau_1(m) - 2\tau_2(m) - (d, \ln\tau(m))}{\|\ln\tau(m)\|} = 0.$$

We have thereby proved the inequality $\beta^{(3)}(d) \geq 0$.

Let us now prove the opposite inequality $\beta^{(3)}(d) \leq 0$. By using constructions similar to those in Lemma 1 in [3], one can show that there exists a sequence $\{t(m)\} \uparrow +\infty$ realizing the upper limit $\beta^{(3)}(d)$ and satisfying the condition $t_j(m) \to +\infty$ as $m \to +\infty$, $j = 1, 2$. Therefore, without loss of generality, we can assume that

$$|(d, \ln t(m))| \leq \|t(m)\|^{2/3}, \qquad m \in N, \tag{32_1}$$

$$t_2(m) \geq \max\{\zeta_0, 128\}, \qquad m \in N, \tag{32_2}$$

and, in the case of the curve $D^{(3)}$ of the form (3_3),

$$\left(\Delta^{(3)}(0,0) - d, \left(\sqrt[3]{t_2(m)}, 1\right)\right) \leq 0, \qquad m \in N. \tag{32_3}$$

Again without loss of generality, we assume that the $t(m)$ belong to strips of the quadrant $R^2_{>1}$ with distinct indices l_m, $1 < l_m < l_{m+1} \to +\infty$ as $m \to +\infty$, and $d \in D^{(3)}(l_m)$, $m \in N$. If the sequence $\{t(m)\}$ has an infinite subsequence $\{t(m_j)\}$ such that $\ln\psi(t(m_j)) - (d, \ln t(m_j)) \leq 0$, then the desired inequality $\beta^{(3)}(d) \leq 0$ is obvious in view of the estimate $\ln\varphi(t(m)) - t_1(m) - 2t_2(m) \leq 0$. Therefore, we assume that

$$\ln\psi(t(m)) - (d, \ln t(m)) > 0 \qquad \forall m \in N. \tag{33}$$

We take some $m \in N$ and suppose that $t(m)$ belongs to the strip $\Pi^{(3)}(l_m)$ used in the realization of the left boundary upper exponent set $\bar{D}_x(\lambda') = D^{(3)}$. Suppose that the curve $D^{(3)}$ has the form (1_3) or (2_3). If $t(m)$ lies in the "main" strip $\Pi^{(3)}(i_m, l_m)$, then from (33), we have $\ln \psi^{(3)}_{i_m,l_m}(t(m)) - (d, \ln t(m)) > 0$, which, together with (14_2), implies that $t(m) \in S\Pi^{(3)}(i_m, l_m)$. Now it follows from (12_2) that $\left|\ln t_2(m)/\ln t_1(m) - \Theta^{(3)}_{i_m,l_m}\right| \le \Delta_2(l)$. Since the curve $D^{(3)}$ is convex, it follows that the point $d \in D^{(3)}$ does not lie below the tangent to the curve $D^{(3)}$ at the point $\Delta^{(3)}(i_m, l_m) \in D^{(3)}$. Therefore, $\Delta^{(3)}_1(i_m, l_m) - d_1 + \Theta^{(3)}_{i_m,l_m}\left(\Delta^{(3)}_2(i_m, l_m) - d_2\right) \le 0$ and

$$\begin{aligned} R_x((1,2), d, t(m)) &= \ln \bar{E}(t(m)) - t_1(m) - 2t_2(m) + \ln \psi^{(3)}_{i_m,l_m}(t(m)) - (d, \ln t(m)) \\ &\le -\ln\left(e^{t_2(m)-t_1(m)} + 1\right) + \left(\Delta^{(3)}(i_m, l_m) - d, \ln t(m)\right) \\ &\le \Bigg[\left\{\left(\Delta^{(3)}_1(i_m, l_m) - d_1\right) + \Theta^{(3)}_{i_m,l_m}\left(\Delta^{(3)}_2(i_m, l_m) - d_2\right)\right\} \\ &\quad + \left(\Delta^{(3)}_2(i_m, l_m) - d_2\right)\left(\frac{\ln t_2(m)}{\ln t_1(m)} - \Theta^{(3)}_{i_m,l_m}\right)\Bigg] \ln t_1(m) \\ &\le \left|\Delta^{(3)}_2(i_m, l_m) - d_2\right| \left|\frac{\ln t_2(m)}{\ln t_1(m)} - \Theta^{(3)}_{i_m,l_m}\right| \ln t_1(m) \le 2^{-l_m} \|\ln t(m)\|. \end{aligned}$$

If $t(m)$ belongs to the transition strip $P^{(3)}(i_m + 1, l_m)$, $i_m \in I^1_{l_m}$, and $\psi^{(3)}_{i_m+1,l_m}(t(m)) \le \psi^{(3)}_{i_m,l_m}(t(m))$, then from the inequality $\psi(t(m)) \le \psi^{(3)}_{i_m,l_m}(t(m))$, one can obtain the estimate

$$R_x((1,2), d, t(m)) \le 2^{-l_m} \|\ln t(m)\| \tag{34}$$

just as for the case in which $t(m) \in \Pi^{(3)}(i_m, l_m)$. But if $\psi^{(3)}_{i_m+1,l_m}(t(m)) > \psi^{(3)}_{i_m,l_m}(t(m))$, then it follows from (33), (18_1), and (17) that $\ln \psi^{(3)}_{i_m+1,l_m}(t(m)) - (d, \ln t(m)) > 0$. By using the same argument as for the case in which $t(m) \in \Pi^{(3)}(i_m, l_m)$ and by writing out the equation of the tangent to the curve $D^{(3)}$ at the point $\Delta^{(3)}(i_m + 1, l_m)$, we obtain the estimate (34). Now we suppose that the curve $D^{(3)}$ has the form (3_3) and $t(m)$ belongs to the strip $\Pi^{(3)}(l_m)$. If either $\ln t_1(m) \le \sqrt[3]{t_2(m)} \ln t_2(m)$ or the inequalities $\ln t_1(m) > \sqrt[3]{t_2(m)} \ln t_2(m)$ and $\left(\Delta^{(3)}(0,0), \ln t(m)\right) \le \ln u^{(3)}_{l_m}(t(m))$ are valid, then it follows from (18_2) that $\psi(t(m)) \le u^{(3)}_{l_m}(t(m))$, which implies (34). But if $\ln t_1(m) > \sqrt[3]{t_2(m)} \ln t_2(m)$ and $\left(\Delta^{(3)}(0,0), \ln t(m)\right) > \ln u^{(3)}_{l_m}(t(m))$, then from (32_3), we have

$$\begin{aligned} R_x((1,2), d, t(m)) &\le \left(\Delta^{(3)}(0,0) - d, \ln t(m)\right) \\ &\le \left(\Delta^{(3)}(0,0) - d, \left(\sqrt[3]{t_2(m)}, 1\right)\right) \ln t_2(m) \le 0. \end{aligned}$$

We have thereby proved the estimate (34) for $R_x((1,2), d, t(m))$ with $t(m) \in \Pi^{(3)}(l_m)$.

Now let $t(m)$ belong to the strip $\Pi^{(4)}(l_m)$ used in the realization of the right boundary upper exponent set $\bar{D}_x(\lambda'') = D^{(4)}$. Suppose that the curve $D^{(4)}$ has the form (1_4) or (2_4). If $t(m)$ lies in the strip $\Pi^{(4)}(i_m, l_m)$, then it follows from (33), (18_1), and (17) that

$$\ln \psi^{(4)}_{i_m,l_m}(t(m)) - (d, \ln t(m)) > 0;$$

consequently, $t(m) \in S\Pi^{(4)}(i_m, l_m)$. From (11_1), we have the inequality $t_2(m) \ge e t_1(m)$, which, together with (23), (32_1), and (32_2), implies that

$$\begin{aligned} R_x((1,2), d, t(m)) &\le \ln \bar{E}(t(m)) - t_1(m) - 2t_2(m) + \ln \psi(t(m)) + (d, \ln t(m)) \\ &\le -\ln\left(e^{t_2(m)-t_1(m)} + 1\right) + 2\|t(m)\|^{2/3} \le t_2^{2/3}(m)\left(-t_2^{1/3}(m) + 4\sqrt[3]{2}\right)/2 \le 0. \end{aligned} \tag{35}$$

If $t(m)$ lies in the strip $P^{(4)}(i_m+1, l_m)$, $i \in I^1_{l_m}$, then it follows from (18_1), (17), (33), and (14_2) that either $t(m) \in S\Pi^{(4)}(i_m, l_m)$ or $t(m) \in S\Pi^{(4)}(i_m+1, l_m)$, and each of these inclusions, together with (11_1), implies that $t_2(m) \ge e t_1(m)$. Then, just as for the case in which $t(m)$ belongs to the strip $\Pi^{(4)}(i_m, l_m)$, one can prove the estimate (35). But if the curve $D^{(4)}$ has the form (3_4) and $\ln t_2(m) \le \sqrt[3]{t_1(m)} \ln t_1(m)$, then, by using the relation $\psi(t(m)) = u^{(4)}_{l_m}(t(m))$, we obtain (35). If $\ln t_2(m) > \sqrt[3]{t_1(m)} \ln t_1(m)$, then it follows from ($13_1$) that $t_2(m) > t_1(m)^{\sqrt[3]{t_1(m)}} \ge 2e t_1(m)$, which implies (35).

We have thereby proved the estimate (35) for the case in which $t(m)$ belongs to the strip $\Pi^{(4)}(l_m)$.

Now consider the case in which $t(m)$ belongs to the strips $\underline{\Pi}(l_m)$ used for the realization of lower boundary exponent sets. By estimating the quantity $R_x((1,2), d, t(m))$ from above and by using (23), (32_1), and (32_2), we obtain

$$\begin{aligned} R_x((1,2), d, t(m)) &\le \ln \underline{E}(t(m)) + \ln \psi(t(m)) - t_1(m) - 2t_2(m) - (d, \ln t(m)) \\ &\le \ln 2 - t_1(m) - 2t_2(m) + 2\|t(m)\|^{2/3} \le \ln 2 - t_2(m) - \|t(m)\| + 2\|t(m)\|^{2/3} \le 0. \end{aligned} \tag{36}$$

If $t(m)$ belongs to the strip $P^{(4)}(1, l_m)$ lying between the strips $\Pi^{(3)}(l_m)$ and $\Pi^{(4)}(l_m)$, then it follows from (18_3) that at least one of the inequalities $\psi(t(m)) \le \omega^{(3)}_{l_m\times 2^{l_m}, l_m}(t(m))$ and $\psi(t(m)) \le \omega^{(4)}_{1,l_m}(t(m))$ is valid. If the first inequality holds, then, just as for the case in which $t(m)$ belongs to the strip $\Pi^{(3)}(l_m)$, one can prove the estimate (34). In the case of the second inequality, by using the relation $\varphi(t(m)) = \bar{E}(t(m))$ and by following the argument for the case in which $t(m)$ belongs to the strip $\Pi^{(4)}(l_m)$, one obtains the estimate (35).

Now let $t(m)$ belong to the strip $P^{(3)}(1, l_m)$ lying between the strips $\underline{\Pi}(l_m)$ and $\bar{\Pi}(l_m)$. If $t(m) \in P^{(3)}_1(1, l_m)$, then it follows from ($18_4$) that $\psi(t(m)) = \omega^{(2)}_{l_m\times 2^{l_m}, l_m}(t(m))$, and relation ($16_3$) implies that one of the inequalities $\varphi(t(m)) \le \underline{E}(t(m))$ and $\ln \varphi(t(m)) \le \zeta(t(m))/2$ is valid. In the case of the first inequality, arguing as in the case of the inclusion $t(m) \in \underline{\Pi}(l_m)$, we prove (36). But if the second inequality is valid, then, by using (23), (32_1), and (32_2), we obtain

$$\begin{aligned} R_x((1,2), d, t(m)) &\le \zeta(t(m))/2 - t_1(m) - 2t_2(m) + \ln \psi(t(m)) - (d, \ln t(m)) \\ &\le -(t_1(m) + 3t_2(m))/2 + 2\|t(m)\|^{2/3} \le -\|t(m)\|/2 + 2\|t(m)\|^{2/3} \le 0. \end{aligned} \tag{37}$$

Let $t(m) \in P^{(3)}_2(1, l_m)$. Then $\ln \varphi(t(m)) = \zeta(t(m))/2$, and consequently, inequality (37) is valid. But if $t(m)$ belongs to the strip $P^{(3)}_3(1, l_m)$, then $\psi(t(m)) = \omega^{(3)}_{1,l_m}(t(m))$, and at least one of the inequalities $\ln \varphi(t(m)) \le \zeta(t(m))/2$ and $\varphi(t(m)) \le \bar{E}(t(m))$ is valid. In the case of the first inequality, we again obtain the estimate (37), and in the other case, arguing as in the case of the inclusion $t(m) \in \Pi^{(3)}(l_m)$, we prove the estimate (34).

Now let $t(m)$ belong to the strip $P^{(1)}(1, l_m+1)$ lying between the strips $\bar{\Pi}(l_m)$ and $\underline{\Pi}(l_m+1)$. If $t(m) \in P^{(1)}_1(1, l_m+1)$, then it follows from ($18_5$) and ($16_4$) that $\psi(t(m)) = \omega^{(4)}_{l_m\times 2^{l_m}, l_m}(t(m))$ and at least one of the inequalities $\varphi(t(m)) \le \bar{E}(t(m))$ and $\ln \varphi(t(m)) \le \zeta(t(m))/2$ is valid. If the first inequality holds, then we obtain the estimate (35) just as for the case in which $t(m) \in \Pi^{(4)}(l_m)$. In the case of the second inequality, we have the estimate (37). If $t(m)$ belongs to the strip $P^{(1)}_2(1, l_m+1)$, then $\ln \varphi(t(m)) = \zeta(t(m))/2$, which implies (37). But if $t(m) \in P^{(1)}_3(1, l_m+1)$, then $\psi(t(m)) = \omega^{(1)}_{1,l_m+1}(t(m))$ and at least one of the inequalities $\ln \varphi(t(m)) \le \zeta(t(m))/2$ and $\varphi(t(m)) \le \underline{E}(t(m))$ is valid. In the first case, we have (37), and in the second case, we obtain (36) just as for the case in which $t(m)$ belongs to the strip $\underline{\Pi}(l_m)$.

We have thereby proved the inequality $\beta^{(3)}(d) \le 0$ and the first condition in definition (4) of the upper characteristic exponent for a vector d. The second condition in this definition is realized along the above-constructed sequence $\tau(m)$. We have thereby proved that the left boundary upper exponent set $\bar{D}_x(\lambda')$ coincides with the given curve $D^{(3)}$ on the everywhere dense set $D^{(3)}_\infty$. This, together with the continuity of the curves $\bar{D}_x(\lambda')$ and $D^{(3)}$, implies that $D^{(3)} \subset \bar{D}_x(\lambda')$. In the case of the curve $D^{(3)}$ of the form (1_3) or (2_3), we have $\bar{D}_x(\lambda') = D^{(3)}$. Let us show that, in the case of the curve $D^{(3)}$ of the form (3_3), the left boundary upper exponent set $\bar{D}_x(\lambda')$ cannot be larger than this curve as well. We take an arbitrary exponent $d \in \bar{D}_x(\lambda')$. Then from the first condition in definition (4) of the upper characteristic exponent, we have $\Delta_1^{(3)}(0,0) - d_1 \le 0$ along some sequence $\{t(m)\} \uparrow +\infty$ of the form $t(m) = \left(t_2(m)^{3\sqrt[3]{t_2(m)}}, t_2(m)\right)$, $t(m) \in \Pi^{(3)}(l_m)$. Since $\bar{D}_x(\lambda')$ is a closed curve and is strictly monotone decreasing, we have $\bar{D}_x(\lambda') = D^{(3)}$ in this case as well.

5. CONSTRUCTION OF THE EQUATION. BOUNDEDNESS OF THE COEFFICIENTS

The function $x(t) > 0$ constructed in the preceding is a solution of Eq. (1_1) with coefficients

$$a_k(t) = \frac{1}{x(t)}\frac{\partial x(t)}{\partial t_k} = \frac{\partial \ln x(t)}{\partial t_k}, \qquad t \in R^2_{>1}, \qquad k = 1,2,$$

satisfying the total integrability condition in view of the infinite differentiability of $\ln x(t)$ in $R^2_{>1}$. Let us prove the boundedness of the coefficients.

We first show that the derivatives $\dfrac{\partial}{\partial t_k}\ln\varphi(t)$, $k = 1,2$, are bounded. Obviously, we have the estimates

$$\left|\frac{\partial}{\partial t_k}\ln \underline{E}(t)\right| = \left|\frac{e^{-t_k}}{e^{-t_1}+e^{-t_2}}\right| \le 1, \qquad t \in R^2_{>1}, \tag{38_1}$$

$$\left|\frac{\partial}{\partial t_k}\ln \bar{E}(t)\right| = \left|1 + \frac{e^{-t_k}}{e^{-t_1}+e^{-t_2}}\right| \le 2, \qquad t \in R^2_{>1}, \tag{38_2}$$

$$\left|\frac{\partial}{\partial t_k}\left(\frac{\zeta(t)}{2}\right)\right| = \frac{1}{2}, \qquad t \in R^2_{>1}. \tag{38_3}$$

From inequality (L_1) in the lemma, we obtain the estimates

$$\left|\frac{\zeta(t)}{2}\frac{\partial}{\partial t_k}e_{01}(\ln\zeta(t);\xi_1,\xi_2)\right| \le 2,$$

$$\left|\ln \underline{E}(t)\frac{\partial}{\partial t_k}e_{01}(\ln\zeta(t);\xi_1,\xi_2)\right| \le \frac{4\ln 2}{\zeta(t)} \le 2\ln 2,$$

$$\left|\ln \bar{E}(t)\frac{\partial}{\partial t_k}e_{01}(\ln\zeta(t);\xi_1,\xi_2)\right| \le 4 + 2\ln 2, \qquad t \in R^2_{>1},$$

on each interval $[\xi_1,\xi_2]$ of length $\xi_2 - \xi_1 \ge 2$. These estimates, together with (16_1)–(16_5) and inequalities (38_1)–(38_3), imply that the derivatives $\dfrac{\partial}{\partial t_k}\ln\varphi(t)$, $k = 1,2$, are bounded.

Now let us show that the derivatives $\dfrac{\partial}{\partial t_k}\ln\psi(t)$, $k = 1,2$, are also bounded. We first estimate the partial derivatives $\dfrac{\partial}{\partial t_k}\ln\psi^{(q)}_{i,l}(t)$, $q = 1,2,3,4$, $k = 1,2$, in the strip $\Pi L^{(q)}(i,l)$, $i \in I_l$, $l \in N$. Taking account of $(8_{1,1})$–$(8_{2,2})$, we note that the inner product $\left(\Delta^{(q)}(i,l), \ln t\right)$ occurs in the definition of the function $\ln\psi^{(q)}_{i,l}(t)$ only in the sector $\tilde{S}^{(q)}(i,l)$ defined as follows. If the slope $k^{(q)}(i,l)$ is not equal to -1, then we set

$$\tilde{S}^{(q)}(i,l) \equiv \left\{t \in R^2_{>1}:\ \left|\frac{\ln t_2}{\ln t_1} - \Theta^{(q)}_{i,l}\right| \le \tau^{(q)}_l + \frac{1}{4}\right\}, \qquad q = 1,4,$$

$$\tilde{S}^{(q)}(i,l) \equiv \left\{t \in R^2_{>1}:\ \left|\frac{\ln t_2}{\ln t_1} - \Theta^{(q)}_{i,l}\right| \le \tau^{(q)}_l + \frac{\Omega^{(q)}_l}{4}\right\}, \qquad q = 2,3.$$

If $k^{(q)}(i,l) = -1$, then we set $\tilde{S}^{(q)}(i,l) \equiv \{t \in R^2_{>1} : \ e-2 \le t_2/t_1 \le 4e\}$ for $q = 1,4$ and $\tilde{S}^{(q)}(i,l) \equiv \{t \in R^2_{>1} : \ 1/(4e) \le t_2/t_1 \le 1/e + 1/2\}$ for $q = 2,3$. Since $\zeta(t) \ge \mu_l^{(q)} \exp(\exp(2))$ in the strip $\Pi L^{(q)}(i,l)$, $i \in I_l$, $l \in N$, $q = 1,2,3,4$, we have the estimates

$$t_k \ge \nu_l^{(q)}, \qquad k = 1,2, \tag{39}$$

in the intersection $\tilde{S}^{(q)}(i,l) \cap \Pi L^{(q)}(i,l)$, $i \in I_l$, $l \in N$, $q = 1,2,3,4$. It was proved in [2] that the partial derivatives of the function $\ln \psi_{i,l}^{(q)}(t)$, $q = 1,4$, given by $(8_{1,1})$ and (9_1) are bounded in the strip $\Pi L^{(q)}(i,l)$, $i \in I_l$, $l \in N$, $q = 1,2,3,4$.

Let us now estimate the partial derivatives of the function $\ln \psi_{i,l}^{(q)}(t)$, $q = 2,3$, given by $(8_{2,1})$ and (9_2). From the partition of the quadrant $R^2_{>1}$, inequalities (39), and the relation $\nu_l^{(q)} \ge 1$ for $t \in \tilde{S}^{(q)}(i,l) \cap \Pi L^{(q)}(i,l)$, $i \in I_l$, $l \in N$, $q = 1,2,3,4$, we obtain the estimates

$$\left| \frac{\partial}{\partial t_k} \left(\Delta^{(q)}(i,l), \ln t\right) \right| = \frac{\left|\Delta_k^{(q)}(i,l)\right|}{t_k} \le \frac{\Delta_1(l)}{\nu_l^{(q)}} \le 1, \qquad k = 1,2. \tag{40}$$

We also have the obvious estimates

$$\frac{\partial}{\partial t_k} \|\ln t\|^2 = \frac{2\ln t_k}{t_k} \le 2, \qquad k = 1,2. \tag{41}$$

By using the first inequality in the lemma, inequality (39), and the relations $\Omega_l^{(q)}/4 \le \ln t_2/\ln t_1 \le 2$ and $\tau_l^{(q)} \le \Omega_l^{(q)}/2$, we obtain the estimates

$$\begin{aligned} &\left| \left(\Delta^{(q)}(i,l), \ln t\right) \frac{\partial}{\partial t_1} \left(1 - e_{01}\left(\frac{\ln t_2}{\ln t_1}; \Theta_{i,l}^{(q)} + \tau_l^{(q)}, \Theta_{i,l}^{(q)} + \tau_l^{(q)} + \frac{\Omega_l^{(q)}}{4}\right)\right) \right| \\ &\quad \le 2 \left\|\Delta^{(q)}(i,l)\right\| (\ln t_1 + \ln t_2) \exp\left[32\left(\Omega_l^{(q)}\right)^{-2}\right] \frac{\ln t_2}{t_1 \ln^2 t_1} \le 12\Delta_1(l) \frac{\exp\left[8\left(\tau_l^{(q)}\right)^{-2}\right]}{\nu_l^{(q)}} \le 1, \end{aligned} \tag{42_1}$$

$$\begin{aligned} &\left| \left(\Delta^{(q)}(i,l), \ln t\right) \frac{\partial}{\partial t_2} \left(1 - e_{01}\left(\frac{\ln t_2}{\ln t_1}; \Theta_{i,l}^{(q)} + \tau_l^{(q)}, \Theta_{i,l}^{(q)} + \tau_l^{(q)} + \frac{\Omega_l^{(q)}}{4}\right)\right) \right| \\ &\quad \le 2\Delta_1(l) (\ln t_1 + \ln t_2) \frac{\exp\left[32\left(\Omega_l^{(q)}\right)^{-2}\right]}{t_2 \ln t_1} \le 1, \end{aligned} \tag{42_2}$$

$$\left| \left(\Delta^{(q)}(i,l), \ln t\right) \frac{\partial}{\partial t_k} e_{01}\left(\frac{\ln t_2}{\ln t_1}; \Theta_{i,l}^{(q)} - \tau_l^{(q)} - \frac{\Omega_l^{(q)}}{4}, \Theta_{i,l}^{(q)} - \tau_l^{(q)}\right) \right| \le 1, \quad k = 1,2, \tag{42_3}$$

$$\begin{aligned} \|\ln t\|^2 \left| \frac{\partial}{\partial t_1} e_{01}\left(\frac{\ln t_2}{\ln t_1}; \Theta_{i,l}^{(q)}, \Theta_{i,l}^{(q)} + \tau_l^{(q)}\right) \right| &\le 2\exp\left[2\left(\tau_l^{(q)}\right)^{-2}\right] \frac{\ln t_2}{t_1 \ln^2 t_1} \left(\ln^2 t_1 + \ln^2 t_2\right) \\ &\le 20 \exp\left[2\left(\tau_l^{(q)}\right)^{-2}\right] \frac{\ln t_1}{t_1} \le 1, \end{aligned} \tag{42_4}$$

$$\begin{aligned} \|\ln t\|^2 \left| \frac{\partial}{\partial t_2} e_{01}\left(\frac{\ln t_2}{\ln t_1}; \Theta_{i,l}^{(q)}, \Theta_{i,l}^{(q)} + \tau_l^{(q)}\right) \right| &\le \frac{2\exp\left[2\left(\tau_l^{(q)}\right)^{-2}\right]}{t_2 \ln t_1} \left(\ln^2 t_1 + \ln^2 t_2\right) \\ &\le 40 \exp\left[2\left(\tau_l^{(q)}\right)^{-2}\right] \frac{\ln t_2}{\Omega_l^{(q)} t_2} \le 1, \end{aligned} \tag{42_5}$$

$$\|\ln t\|^2 \left| \frac{\partial}{\partial t_k} \left(1 - e_{01}\left(\frac{\ln t_2}{\ln t_1}; \Theta_{i,l}^{(q)} - \tau_l^{(q)}, \Theta_{i,l}^{(q)}\right)\right) \right| \le 1, \quad k = 1,2, \tag{42_6}$$

in the intersection $\tilde{S}^{(q)}(i,l) \cap \Pi L^{(q)}(i,l)$, $i \in I_l$, $l \in N$, $q = 2,3$.

The estimates (40), (41), and (42_1)–(42_6) and relations (6) and (7) imply that the partial derivatives of the function $\ln\psi_{i,l}^{(q)}(t)$, $q=2,3$, given by $(8_{2,1})$ and (9_2) are bounded in the strip $\Pi L^{(q)}(i,l)$, $i\in I_l$, $l\in N$, $q=2,3$.

Let us now show that the partial derivatives of the function $\ln\psi_{i,l}^{(q)}(t)$, $q=1,4$, given by $(8_{1,2})$ are bounded in the strip $\Pi L^{(q)}(i,l)$, $i\in I_l$, $l\in N$, $q=1,4$. By using the second inequality in the lemma, we obtain the estimates

$$\left|\left(\Delta^{(q)}(i,l),\ln t\right)\frac{\partial}{\partial t_1}\left(1-e_{01}\left(\frac{t_2}{t_1};3e,4e\right)\right)\right| \le 4\Delta_1(l)\frac{(\ln t_1+\ln t_2)\,t_2}{t_1^2} \le 16e\frac{\Delta_1(l)}{\sqrt{\nu_l^{(q)}}}\frac{2\ln t_1+\ln 4e}{\sqrt{t_1}}\le 1,$$

$$\left|\left(\Delta^{(q)}(i,l),\ln t\right)\frac{\partial}{\partial t_2}\left(1-e_{01}\left(\frac{t_2}{t_1};3e,4e\right)\right)\right| \le 4\Delta_1(l)\frac{\ln t_1+\ln t_2}{t_1}\le 4\frac{\Delta_1(l)}{\sqrt{\nu_l^{(q)}}}\frac{2\ln t_1+\ln 4e}{\sqrt{t_1}}\le 1,$$

$$\left|\left(\Delta^{(q)}(i,l),\ln t\right)\frac{\partial}{\partial t_k}e_{01}\left(\frac{t_2}{t_1};e-2,e\right)\right|\le 1,\quad k=1,2,$$

in the intersection $\tilde S^{(q)}(i,l)\cup\Pi L^{(q)}(i,l)$, which, together with the obvious estimates

$$\|\ln t\|^2\left|\frac{\partial}{\partial t_k}e_{101}\left(\frac{t_2}{t_1};e,2e,3e\right)\right|\le 1,\quad t\in\tilde S^{(q)}(i,l)\cap\Pi L^{(q)}(i,l),\quad k=1,2,$$

imply that the partial derivatives of the function $\ln\psi_{i,l}^{(q)}(t)$, $q=1,4$, given by $(8_{1,2})$ are bounded in the strip $\Pi L^{(q)}(i,l)$, $i\in I_l$, $l\in N$, $q=1,4$.

In a similar way, by using the first inequality in the lemma, one can show that the partial derivatives of the function $\ln\psi_{i,l}^{(q)}(t)$, $q=2,3$, given by $(8_{2,2})$ are bounded in the strip $\Pi L^{(q)}(i,l)$, $i\in I_l$, $l\in N$, $q=2,3$.

In view of (17), we have thereby shown that the partial derivatives of the function $\ln u_l^{(q)}(t)$, $q=1,2,3,4$, are bounded in each "main" strip $\Pi^{(q)}(i,l)$, $i\in I_l$, $l\in N$, $q=1,2,3,4$. Again in view of (17), to prove the boundedness of these partial derivatives in the "transition" strip $P^{(q)}(i+1,l)$, $i\in I_l^1$, $l\in N$, $q=1,2,3,4$, it suffices to justify the boundedness of the products

$$\left[\ln\psi_{i+1,l}^{(q)}(t)-\ln\psi_{i,l}^{(q)}(t)\right]\frac{\partial}{\partial t_k}e_{01}\left(\ln\zeta(t);\ln\alpha_{i+1,l}^{(q)},\ln\beta_{i+1,l}^{(q)}\right),\quad k=1,2.$$

Since $\ln\beta_{i+1,l}^{(q)}-\ln\alpha_{i+1,l}^{(q)}=\ln e^6=6>2$ and $\zeta(t)\ge\nu_l^{(q)}\ge c$ for all $t\in P^{(q)}(i+1,l)$, it follows from the second inequality in the lemma that

$$\left|\ln\psi_{j,l}^{(q)}(t)\frac{\partial}{\partial t_k}e_{01}\left(\ln\zeta(t);\ln\alpha_{i+1,l}^{(q)},\ln\beta_{i+1,l}^{(q)}\right)\right| \le 4\frac{\Delta_1(l)\,(\ln t_1+\ln t_2)\,\chi_{\tilde S^{(q)}(j,l)}(t)+\|\ln t\|^2}{\zeta(t)}\le 1,$$

where $\chi_{\tilde S^{(q)}(j,l)}(t)$ is the characteristic function of the set $\tilde S^{(q)}(j,l)$ and $j=i,i+1$. These estimates, together with the boundedness of the partial derivatives $\dfrac{\partial}{\partial t_k}\ln\psi_{i,l}^{(q)}(t)$ and $\dfrac{\partial}{\partial t_k}\ln\psi_{i+1,l}^{(q)}(t)$ in the "transition" strip $P^{(q)}(i+1,l)$, imply that the partial derivatives $\dfrac{\partial}{\partial t_k}\ln u_l^{(q)}(t)$ are also bounded in this strip.

We have thereby proved that, by definition (18_1) of the function $\psi(t)$, the partial derivatives $\dfrac{\partial}{\partial t_k}\ln\psi(t)$, $k=1,2$, are bounded in the strip $\Pi^{(q)}(l)$, $l\in N$, $q=1,2,3,4$, in the case of the curve $D^{(q)}$ of the form (1_q) or (2_q).

Let us now prove the boundedness of the derivatives $\dfrac{\partial}{\partial t_k}\ln\psi(t)$, $k=1,2$, in the strip $\Pi^{(q)}(l)$, $l\in N$, $q=1,2,3,4$, for the case in which the curve $D^{(q)}$ has the form (3_q). In each strip $\Pi^{(q)}(i,l)$, $i\in I_l$, $l\in N$, $q=1,4$, we have the estimates

$$
\begin{aligned}
\left|\left(\Delta^{(q)}(0,0),\ln t\right)\frac{\partial\chi^{(q)}(t)}{\partial t_1}\right| &\le 4\left\|\Delta^{(q)}(0,0)\right\|(\ln t_1+\ln t_2)\frac{\ln t_2}{3t_1^{4/3}\ln^2 t_1}(\ln t_1+3)\\
&\le 4\left\|\Delta^{(q)}(0,0)\right\|\frac{\ln t_1+3+3\sqrt[3]{t_1}\ln t_1+9\sqrt[3]{t_1}}{t_1}\le\left\|\Delta^{(q)}(0,0)\right\|b,\\
\left|\left(\Delta^{(q)}(0,0),\ln t\right)\frac{\partial\chi^{(q)}(t)}{\partial t_2}\right| &\le 4\left\|\Delta^{(q)}(0,0)\right\|\frac{\ln t_1+\ln t_2}{\sqrt[3]{t_1}t_2\ln t_1}\le 4\left\|\Delta^{(q)}(0,0)\right\|\left(\frac{1}{\sqrt[3]{t_1}t_2}+\frac{3}{t_2}\right)\\
&\le\left\|\Delta^{(q)}(0,0)\right\|b,\\
\left|\ln\psi_{i,l}^{(q)}(t)\frac{\partial\chi^{(q)}(t)}{\partial t_1}\right| &\le 4\left(\Delta_1(l)(\ln t_1+\ln t_2)\chi_{\tilde S^{(q)}(i,l)}(t)+\|\ln t\|^2\right)\frac{\ln t_2}{3t_1^{4/3}\ln^2 t_1}(\ln t_1+3)\\
&\le 4\frac{\Delta_1(l)}{t_1}\chi_{\tilde S^{(q)}(i,l)}(t)\left(\ln t_1+3+3\sqrt[3]{t_1}\ln t_1+9\sqrt[3]{t_1}\right)\\
&\quad+\frac{4\left(\ln^2 t_1+3\ln t_1+9\left(\sqrt[3]{t_1}\right)^2\ln^2 t_1+27\left(\sqrt[3]{t_1}\right)^2\ln t_1\right)}{t_1}\\
&\le\chi_{\tilde S^{(q)}(i,l)}(t)+b,\\
\left|\ln\psi_{i,l}^{(q)}(t)\frac{\partial\chi^{(q)}(t)}{\partial t_2}\right| &\le\frac{4\left(\Delta_1(l)(\ln t_1+\ln t_2)\chi_{\tilde S^{(q)}(i,l)}(t)+\|\ln t\|^2\right)}{\sqrt[3]{t_1}t_2\ln t_1}\\
&\le 4\left(\Delta_1(l)\chi_{\tilde S^{(q)}(i,l)}(t)\left(\frac{1}{\sqrt[3]{t_1}t_2}+\frac{3}{t_2}\right)+\frac{\ln t_1}{\sqrt[3]{t_1}}+\frac{3\ln t_2}{t_2}\right)\\
&\le\chi_{\tilde S^{(q)}(i,l)}(t)+b
\end{aligned}
$$

with some constant $b>0$. Similar estimates can be proved for $q=2,3$. These estimates, together with the boundedness of the derivatives $\dfrac{\partial}{\partial t_k}\ln u_l^{(q)}(t)$ in the strip $\Pi^{(q)}(l)$, $l\in N$, $q=1,2,3,4$, imply that the derivatives of the function $\ln\psi(t)$ are bounded in this strip for the case in which the curve $D^{(q)}$ has the form (3_q).

Taking account of the definition of the "transition" strips $P^{(q)}(1,l)$, $l\in N$, $q=1,2,3,4$, and using the same considerations as for the strips $P^{(q)}(i+1,l)$, $i\in I_l^1$, $l\in N$, $q=1,2,3,4$, one can show that the derivatives $\dfrac{\partial}{\partial t_k}\ln\psi(t)$, $k=1,2$, are also bounded in these strips.

We have thereby proved that the coefficients $a_i(t)$, $i=1,2$, are bounded in the entire quadrant $R^2_{>1}$. The proof of Theorem 1 is complete.

Theorem 1 implies that the boundary exponent sets of a solution can have an arbitrary relative position, unlike characteristics sets, whose arrangement is subjected to the condition [4]

$$\sup\{p_i : p\in P_x\}\le\inf\{\lambda_i :\ \lambda\in\Lambda_x\},\qquad i=1,2.$$

The following assertion provides a complete joint description of boundary exponent sets.

Theorem 2. *Given sets $D^{(q)}$, $q=1,2,3,4$, are the left and right boundary lower exponent sets $\underline{D}_x(p')$ and $\underline{D}_x(p'')$ and the left and right boundary upper exponent sets $\bar{D}_x(\lambda')$ and $\bar{D}_x(\lambda'')$, respectively, of some nontrivial solution $x(t)$, whose lower characteristic set P_x and characteristic set*

Λ_x *are assumed to consist of more than one point, of some completely integrable Pfaff system* (1) *with bounded continuously differentiable coefficients if and only if the sets* $D^{(1)}$ *and* $D^{(2)}$ (*respectively, the sets* $D^{(3)}$ *and* $D^{(4)}$) *either are empty or can be represented in the form of closed upper* (*respectively, lower*) *convex monotone decreasing curves on a two-dimensional plane which are unbounded on the right and below* (*respectively, on the left and above*) *and whose slope at each interior point is negative and is not less* (*for* $D^{(1)}$ *and* $D^{(4)}$) *or not greater* (*for* $D^{(2)}$ *and* $D^{(3)}$) *than* -1.

Remark. The set $I_l = \{0, 1, \ldots, l \times 2^l\}$ used in [2] should have the form $I_l = \{1, \ldots, l \times 2^l\}$.

REFERENCES

1. Izobov, N.A. and Krupchik, E.N., *Differents. Uravn.*, 2003, vol. 39, no. 3, pp. 308–319.
2. Izobov, N.A. and Krupchik, E.N., *Differents. Uravn.*, 2002, vol. 38, no. 10, pp. 1310–1321.
3. Izobov, N.A. and Krupchik, E.N., *Differents. Uravn.*, 2001, vol. 37, no. 5, pp. 616–627.
4. Izobov, N.A., *Differents. Uravn.*, 1997, vol. 33, no. 12, pp. 1623–1630.