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ORDINARY DIFFERENTIAL EQUATIONS

A Description of Exponent Sets of a Solution of a Pfaffian Linear System with Trivial Characteristic Sets

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We consider the Pfaffian linear system

$$\partial x/\partial t_i = A_i(t)x, \qquad x \in R^n, \qquad t = (t_1, t_2) \in R^2_{>1}, \qquad i = 1, 2, \tag{1}$$

with bounded continuously differentiable matrix functions $A_i(t)$ satisfying the complete integrability condition [1, pp. 43–44; 2, pp. 21–24]

$$\partial A_1(t)/\partial t_2 + A_1(t)A_2(t) = \partial A_2(t)/\partial t_1 + A_2(t)A_1(t), \qquad t \in R^2_{>1}.$$

Let $\Lambda_x = \{\lambda[x]\}$ and $P_x = \{p[x]\}$ be the characteristic set [3] and the lower characteristic set [4], respectively, of a nontrivial solution $x : R^2_{>1} \to R^n \backslash \{0\}$ of system (1). On the basis of the definition of the Demidovich characteristic exponent [5] of a solution of an ordinary differential system, the upper characteristic exponent $\bar{d} = \bar{d}_x(\lambda) \in R^2$ and the lower characteristic exponent $\underline{d} = \underline{d}_x(p) \in R^2$ of the solution x were introduced in [6, 7] for a characteristic vector $\lambda \in \Lambda_x$ and a lower characteristic vector $p \in P_x$; they are given by the conditions

$$\begin{aligned} \overline{\ln}_x\left(\lambda, \bar{d}\,\right) \equiv \varlimsup_{t\to\infty} \frac{\ln\|x(t)\| - (\lambda, t) - \left(\bar{d}, \ln t\right)}{\|\ln t\|} = 0, \\ \ln t \equiv (\ln t_1, \ln t_2) \in R^2_+, \\ \overline{\ln}_x\left(\lambda, \bar{d} - \varepsilon e_i\right) > 0 \quad \forall \varepsilon > 0, \quad i = 1, 2, \\ \underline{\ln}_x\left(p, \underline{d}\right) \equiv \varliminf_{t\to\infty} \frac{\ln\|x(t)\| - (p, t) - \left(\underline{d}, \ln t\right)}{\|\ln t\|} = 0, \\ \underline{\ln}_x\left(p, \underline{d} + \varepsilon e_i\right) < 0 \quad \forall \varepsilon > 0, \quad i = 1, 2. \end{aligned} \tag{2}$$

In these papers, the individual upper exponent set $\bar{D}_x(\lambda) = \left\{\bar{d}_x(\lambda)\right\}$ and lower exponent set $\underline{D}_x(p) = \{\underline{d}_x(p)\}$ were also introduced.

By $\lambda', \lambda'' \in \Lambda_x$, $\lambda_1' \le \lambda_1 \le \lambda_1''$ for all $\lambda \in \Lambda_x$ (respectively, $p', p'' \in P_x$, $p_1' \le p_1 \le p_1''$ for all $p \in P_x$) we denote the left and right boundary points, respectively, of the characteristic set Λ_x (respectively, the lower characteristic set P_x) of a solution $x \ne 0$ of system (1).

In the case of a nontrivial characteristic set Λ_x, $\lambda' \ne \lambda''$, and a nontrivial lower characteristic set P_x, $p' \ne p''$, of a solution $x \ne 0$ of system (1), the following results were obtained: the individual interior exponent set $\bar{D}_x(\lambda)$, $\lambda \in (\lambda', \lambda'')$, and the lower exponent sets $\underline{D}_x(p)$, $p \in (p', p'')$, of this solution were completely described in [7]; separate complete descriptions of the left boundary upper exponent set $\bar{D}_x(\lambda')$, the right boundary upper exponent set $\bar{D}_x(\lambda'')$, the left boundary lower exponent set $\underline{D}_x(p')$, and the right boundary lower exponent set $\underline{D}_x(p'')$ were given in [8, 9]; a joint description of all boundary exponent sets of a solution $x \ne 0$ of system (1) was obtained in [10, 11].

As to the trivial characteristic set Λ_x, $\lambda' = \lambda'' \equiv \lambda^0$, and the trivial lower characteristic set P_x, $p' = p'' \equiv p^0$, of a solution $x \ne 0$ of system (1), necessary properties of the upper exponent set $\bar{D}_x \equiv \bar{D}_x(\lambda^0)$ and the lower exponent set $\underline{D}_x \equiv \underline{D}_x(p^0)$ were obtained in [8]; more precisely, it was shown that the nonempty upper exponent set $\bar{D}_x$ (respectively, the lower exponent set $\underline{D}_x$) of this solution is a continuous closed decreasing convex (respectively, concave) curve on the two-dimensional plane. In the present paper, we prove the sufficiency of these necessary properties and hence obtain a complete description of exponent sets of a solution $x \neq 0$ of system (1) with trivial characteristic sets.

Theorem 1. *For each positive integer n, each continuous closed decreasing concave curve D on the two-dimensional plane, and each point $p^0 \in R^2$, there exists a completely integrable n-dimensional system* (1) *with infinitely differentiable bounded coefficients such that every solution $x : R^2_{>1} \to R^n \backslash \{0\}$ of this system has the trivial lower characteristic set $P_x = \{p^0\}$ and the lower exponent set $\underline{D}_x$ coincides with the curve D.*

Proof. Note that to prove this theorem, it suffices to construct a one-dimensional completely integrable Pfaffian equation

$$\partial x/\partial t_i = a_i(t)x, \qquad x \in R, \qquad t \in R^2_{>1}, \qquad i = 1, 2, \tag{1_1}$$

with infinitely differentiable bounded coefficients $a_i(t)$, $i = 1, 2$, and the desired lower characteristic and exponent sets of every nontrivial solution.

We construct the desired equation (1_1) by constructing its nontrivial solution. We define the solution x by the relation $x = \phi\psi$, where $\ln \phi(t) = (p^0, t)$, $t \in R^2_{>1}$. We construct the function ψ so as to ensure that the lower characteristic set P_x of the solution x coincides with the lower characteristic set $P_\phi = \{p^0\}$ of the function ϕ and the lower exponent set $\underline{D}_x$ coincides with the curve D.

It follows from the properties of the curve D that it can have one of the following ten possible forms: it can be

(1) unbounded on the left, on the right, and below and bounded above;
(2) unbounded on the left, above, on the right, and below;
(3) bounded on the left and above and unbounded on the right and below;
(4) unbounded below, on the left, and above and bounded on the right;
(5) bounded below and on the right and unbounded on the left and above;
(6) bounded on the right, below, and above and unbounded on the left;
(7) bounded above, on the left, and on the right and unbounded below;
(8) bounded above and on the right and unbounded below and on the left;
(9) bounded above, below, on the left, and on the right;
(10) a singleton.

1. First, we suppose that the curve D has one of the forms (1)–(9). Then to construct a function ψ implementing the lower exponent set $\underline{D}_x = D$, we introduce the following partition of the curve D.

1.1. Partition of the curve *D*. If the curve D has the form (1) or (2), then its lth partition D_l, $l \in N$, consists of points $\Delta(i,l) \in D$ with the first components

$$\Delta_1(i,l) = \left(i \times 2^{1-l} - l\right)\gamma, \qquad i \in \left\{1, 2, \ldots, l \times 2^l\right\} \equiv I_l;$$

in the case of a curve D of the form (3) with the left boundary point $\Delta' \in D$, its lth partition D_l contains only points $\Delta(i,l) \in D$ with the first components $\Delta_1(i,l) = \Delta'_1 + i\gamma \times 2^{-l}$, $i \in I_l$.

But if the curve D has the form (4), then the lth partition $D_l = \bigcup_{i\in I_l}\{\Delta(i,l)\} \subset D$ consists of the points $\Delta(i,l) \in D$ with the second components $\Delta_2(i,l) = \left(i \times 2^{1-l} - l\right)\gamma$, $i \in I_l$.

In cases (5) and (6) of the curve D with right boundary point $\Delta'' \in D$, as well as in case (8) of the curve D with the vertical asymptote $d_1 = \Delta''_1$, the lth partition D_l of this curve consists of points $\Delta(i,l) \in D$ with the first components $\Delta_1(i,l) = \Delta''_1 - i\gamma \times 2^{-l}$, $i \in I_l$.

In case (7) of the curve D with the left boundary point $\Delta' \in D$, its lth partition D_l contains only points $\Delta(i,l) \in D$ with second components $\Delta_2(i,l) = \Delta_2' - i\gamma \times 2^{-l}$, $i \in I_l$.

If the curve D has one of the forms (1)–(8), then we set $i_l \equiv l \times 2^l$ for the last element of the set I_l.

In case (9) of the curve D with the left boundary point $\Delta' \in D$ and the right boundary point $\Delta'' \in D$, its lth partition D_l consists of points $\Delta(i,l) \in D$ with the first components $\Delta_1(i,l) = \Delta_1' + (\Delta_1'' - \Delta_1')\, i \times 2^{-(l+1)}$, $i \in \{1, 2, \ldots, 2^{l+1} - 1\}$. In this case, the set I_l is defined as $\{1, 2, \ldots, 2^{l+1} - 1\}$, and we set $i_l \equiv 2^{l+1} - 1$.

In each case, by continuing the partition of the curve D infinitely, we obtain a countable set $D_\infty = \bigcup_{l \in N} \bigcup_{i \in I_l} \{\Delta(i,l)\} \subset D$, which is dense everywhere on the curve D.

Note that the partitions D_l of the curve D satisfy the inclusion $D_l \subset D_{l+1}$, $l \in N$.

1.2. Construction of the solution. At each ith point $\Delta(i,l) \in D$, $i \in I_l$, of the lth partition, $l \in N$, we draw some support line $d_2 - \Delta_2(i,l) = k(i,l)\,(d_1 - \Delta_1(i,l))$, $k(i,l) \in (-\infty, 0)$, $(d_1, d_2) \in R^2$, to D that does not lie below D. The existence of such a support line follows from the concavity of D, its decay, and the fact that, by definition, all points $\Delta(i,l)$ of each lth partition D_l are interior points of this curve. At each point $d \in D_\infty$ of an arbitrary finite partition of D, we draw the same support line to prove the existence of a sequence implementing the limit $\underline{\ln}_x\,(p^0, d)$ occurring in the definition of the lower characteristic exponent.

We introduce the following notation:

$$\Theta_{i,l} \equiv 1/|k(i,l)|, \qquad i \in I_l, \qquad \Theta_l \equiv \max_{i \in I_l} \{\Theta_{i,l}\}, \qquad \Omega_l \equiv \min_{i \in I_l} \{\Theta_{i,l}\},$$

$$\Delta_1(l) \equiv \max_{i \in I_l} \{\|\Delta(i,l)\|\}, \qquad \Delta_2(l) \equiv 2^{-l}\, \|\Delta(i_l, l) - \Delta(1,l)\|^{-1}, \qquad l \in N.$$

To match different infinitely differentiable functions with the preservation of their infinite differentiability, we use the infinitely differentiable functions

$$e_{101}(\tau; \alpha_1, \alpha_2, \alpha_3) = e_{01}(\tau; \alpha_2, \alpha_3) + [1 - e_{01}(\tau; \alpha_1, \alpha_2)],$$
$$e_{0110}(\tau; \alpha_1, \alpha_2, \alpha_3, \alpha_4) = e_{01}(\tau; \alpha_1, \alpha_2)\,(1 - e_{01}(\tau; \alpha_3, \alpha_4)),$$

$\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$, $\tau \in R$, constructed on the basis of the function [12]

$$e_{01}(\tau; \tau_1, \tau_2) = \begin{cases} \exp\left\{-(\tau - \tau_1)^{-2} \exp\left[-(\tau_2 - \tau)^{-2}\right]\right\} & \text{for} \quad \tau \in (\tau_1, \tau_2) \\ [1 + \operatorname{sgn}(\tau - 2^{-1}(\tau_1 + \tau_2))]/2 & \text{for} \quad \tau \notin (\tau_1, \tau_2), \end{cases}$$
$$-\infty < \tau_1 < \tau_2 < +\infty.$$

We introduce the functions

$$\ln \psi_{i,l}(t) \equiv (\Delta(i,l), \ln t) e_{0110}\left(\frac{\ln t_2}{\ln t_1}; \Theta_{i,l} - \frac{5\tau_l}{4}, \Theta_{i,l} - \tau_l, \Theta_{i,l} + \tau_l, \Theta_{i,l} + \frac{5\tau_l}{4}\right)$$
$$+ \|\ln t\|^2 e_{101}\left(\frac{\ln t_2}{\ln t_1}; \Theta_{i,l} - \tau_l, \Theta_{i,l}, \Theta_{i,l} + \tau_l\right), \qquad t \in R^2_{>1}, \qquad i \in I_l, \qquad l \in N,$$
$$\tau_l \equiv \min\left\{\frac{1}{2}; \frac{\Omega_l}{2}, \Delta_2(l)\right\}.$$

It follows from the form of the function $\psi_{i,l}$ that there exists a number $T_l \geq 1$ such that

$$\ln \psi_{i,l}(t) - (d, \ln t) \geq 0, \qquad t \in R^2_{>1} \backslash S(i,l),$$
$$S(i,l) \equiv \left\{t \in R^2_{>1} : \left|\frac{\ln t_2}{\ln t_1} - \Theta_{i,l}\right| \leq \tau_l\right\},$$
$$\|t\| \geq T_l, \qquad d \in D_l, \qquad i \in I_l.$$

We split the domain of the solution $x : R^2_{>1} \to R\backslash\{0\}$ to be constructed into disjoint strips by the lines $\zeta(t) \equiv t_1 + t_2 = \text{const}$. To this end, on the basis of some values $\eta_1 \geq T_1$ and $c \geq \exp(100)$, we introduce the numbers

$$\nu_l = c\left(\Theta_l^6 + \Omega_l^{-2}\right)\left(\Delta_1^2(l) + 1\right)\exp\left(c\tau_l^{-2}\right), \qquad \alpha_{i,l} = \left(\eta_l + \nu_l^{4(\Theta_l + \Omega_l^{-1})}\right)\exp(\exp i),$$

$\beta_{i,l} = e^2\alpha_{i,l}$, $i \in I_l$, and $\eta_{l+1} = \beta_{i_l,l} + T_{l+1} + 2^l$, $l \in N$.

We introduce the "basic" strips

$$\Pi(i,l) = \{t \in R^2_{>1} :\ \beta_{i,l} \leq \zeta(t) \equiv t_1 + t_2 \leq \alpha_{i+1,l}\}, \qquad i \in I_l\backslash\{i_l\} \equiv I_l^1,$$
$$\Pi(i_l,l) = \{t \in R^2_{>1} :\ \beta_{i_l,l} \leq \zeta(t) \leq \alpha_{1,l+1}\}, \qquad l \in N,$$

the "transition" strips $P(i,l) = \{t \in R^2_{>1} : \alpha_{i,l} < \zeta(t) < \beta_{i,l}\}$, $i \in I_l$, $l \in N$, and the triangle $T = \{t \in R^2_{>1} :\ \zeta(t) \leq \alpha_{1,1}\}$.

Let us proceed to the construction of the function ψ used in the implementation of the desired lower exponent set $\underline{D}_x = D$. We first introduce the auxiliary function $\tilde{\psi}$ given by the relations

$$\ln\tilde{\psi}(t) = \ln\psi_{i,l}(t) + [\ln\psi_{i+1,l}(t) - \ln\psi_{i,l}(t)]\, e_{01}\left(\ln\zeta(t); \ln\alpha_{i+1,l}, \ln\beta_{i+1,l}\right),$$
$$t \in \Pi(i,l) \cup P(i+1,l) \cup \Pi(i+1,l), \quad i \in I_l^1, \quad l \in N,$$
$$\ln\tilde{\psi}(t) = \ln\psi_{i_l,l}(t) + [\ln\psi_{1,l+1}(t) - \ln\psi_{i_l,l}(t)]\, e_{01}\left(\ln\zeta(t); \ln\alpha_{1,l+1}, \ln\beta_{1,l+1}\right),$$
$$t \in P(1,l+1), \quad l \in N,$$
$$\ln\tilde{\psi}(t) = \ln\psi_{1,1}(t) e_{01}\left(\ln\zeta(t); \ln\alpha_{1,1}, \ln\beta_{1,1}\right), \quad t \in T \cup P(1,1).$$

In the case of the curve D of one of the forms (1), (2), (4), and (8), we set $\psi(t) = \tilde{\psi}(t)$, $t \in R^2_{>1}$. In the case of the curve D of the form (3) or (7) with the left boundary point $\Delta' \in D$, we define the function ψ by the formula

$$\ln\psi(t) = \ln\tilde{\psi}(t) + \left[(\Delta', \ln t) - \ln\tilde{\psi}(t)\right] e_{01}\left(\frac{\ln t_2}{\sqrt[3]{t_1}\ln t_1}; 1, 3\right), \quad t \in R^2_{>1}. \tag{3}$$

In the case of the curve D of the form (5) or (6) with the right boundary point $\Delta'' \in D$, we set

$$\ln\psi(t) = \ln\tilde{\psi}(t) + \left[(\Delta'', \ln t) - \ln\tilde{\psi}(t)\right] e_{01}\left(\frac{\ln t_1}{\sqrt[3]{t_2}\ln t_2}; 1, 3\right), \quad t \in R^2_{>1}.$$

Finally, in the case of the curve D of the form (9) with the left boundary point $\Delta' \in D$ and the right boundary point $\Delta'' \in D$, we define the function ψ by the formula

$$\ln\psi(t) = \ln\tilde{\psi}(t) + \left[(\Delta', \ln t) - \ln\tilde{\psi}(t)\right] e_{01}\left(\frac{\ln t_2}{\sqrt[3]{t_1}\ln t_1}; 1, 3\right)$$
$$+ \left[(\Delta'', \ln t) - \ln\tilde{\psi}(t)\right] e_{01}\left(\frac{\ln t_1}{\sqrt[3]{t_2}\ln t_2}; 1, 3\right), \quad t \in R^2_{>1}.$$

1.3. Construction of the equation. Boundedness of the coefficients. The function $x > 0$ constructed above is a solution of Eq. (1_1) with coefficients given by the relation

$$a_k(t) = x^{-1}(t)\partial x(t)/\partial t_k = \partial\ln x(t)/\partial t_k, \qquad t \in R^2_{>1}, \qquad k = 1, 2. \tag{4}$$

Since $\ln x$ is infinitely differentiable in $R^2_{>1}$, it follows that these coefficients satisfy the complete integrability condition. To prove their boundedness, it suffices to show that the derivatives $\partial\ln\phi(t)/\partial t_k$ and $\partial\ln\psi(t)/\partial t_k$, $k = 1, 2$, are bounded. Obviously, the derivatives $\partial\ln\phi(t)/\partial t_k$, $k = 1, 2$, are bounded; by using the construction of the partition of the quadrant $R^2_{>1}$ and considerations similar to the proof in [9, item 5], one can show that the derivatives $\partial \ln \psi(t)/\partial t_k$, $k = 1, 2$, are bounded.

1.4. Construction of the lower characteristic set. Following item 3 in [9], one can show that the lower characteristic set P_x of the solution x of Eq. (1_1) coincides with the lower characteristic set $P_\phi = \{p^0\}$ of the function ϕ.

1.5. Proof of the coincidence of the lower exponent set with the given curve *D*. We take an arbitrary point d of the everywhere dense partition D_∞ of the curve D and show that it belongs to the lower exponent set $\underline{D}_x$ of the solution x.

By $\beta(d) \equiv \underline{\lim}_{t\to\infty}[(\ln \psi(t) - (d, \ln t))/\|\ln t\|]$ we denote the limit occurring in definition (2) of the lower characteristic exponent of a solution x and show that $\beta(d) = 0$.

Since the point d belongs to the countable partition D_∞ of the curve D and the inclusion $D_l \subset D_{l+1}$, $l \in N$, is valid for each finite partition D_l, it follows that there exists an index $l(d) \in N$ such that $d \in D_l$ for all $l \geq l(d)$ and $d \notin D_l$ for all $l < l(d)$. Let d be the i_mth point of the $(l(d) + m)$th partition; i.e., $d = \Delta(i_m, l(d) + m)$, $m \in N$. Then in each strip $\Pi(i_m, l(d) + m)$, $m \in N$, we take a point $\tau(m)$ on the curve $(\ln t_2)/\ln t_1 = 1/|k(d)|$. The sequence $\{\tau(m)\} \uparrow \infty$ thus defined satisfies the relation $\ln \psi(\tau(m)) = (d, \ln \tau(m))$ for sufficiently large $m \in N$ and $\beta(d) \leq 0$.

Let us now prove the inequality $\beta(d) \geq 0$. By $\{t(m)\} \uparrow \infty$ we denote a sequence realizing the limit $\beta(d)$. Without loss of generality, we assume that all elements $t(m)$ of this sequence belong to strips on $R^2_{>1}$ with different indices l_m, i.e., $t(m) \in P(i_m, l_m) \cup \Pi(i_m, l_m)$ with some $i_m \in I_{l_m}$ and $1 < l_m < l_{m+1} \to \infty$ as $m \to \infty$; moreover, $d \in D_{l_m}$, $m \in N$. By performing considerations similar to those in [9, item 4], we obtain the estimate

$$\ln \tilde{\psi}(t(m)) - (d, \ln t(m)) \geq -2^{-l_m}\|\ln t(m)\|, \qquad m \in N, \tag{5}$$

since the condition $p' \neq p''$ as well as the inequality $k(i, l) \geq -1$ for the angular coefficient of the support line at a point of the partition $\Delta(i, l)$, $i \in I_l$, $l \in N$, of the curve D were not used in the proof of this estimate [9]; these relations were valid in Theorem 1 in [9] but fail in the present theorem.

In the case of a curve D of one of the forms (1), (2), (4), and (8), the relation $\psi(t(m)) = \tilde{\psi}(t(m))$ and inequality (5) imply the estimate $\beta(d) \geq \lim_{m\to\infty}\left(-2^{-l_m}\right) = 0$. We have thereby proved the first condition $\underline{\ln}_x(p^0, d) = 0$ in definition (2) of the lower characteristic exponent for a point $d \in D_\infty$. The second condition in this definition can be proved on the basis of the above-constructed sequence $\{\tau(m)\}$. We have thereby proved the inclusion $D_\infty \subset \underline{D}_x(p^0) = \underline{D}_x$, which, together with the density of the set D_∞ everywhere on the curve D and the continuity of the curves D and $\underline{D}_x$, implies that the curve D belongs to the set $\underline{D}_x$. From the form of the curve D and necessary properties of the lower exponent set $\underline{D}_x$, we obtain the relation $\underline{D}_x = D$.

Now let the curve D have the form (3) or (7). Then, without loss of generality, we restrict our considerations to the following three possibilities:

(1) $\ln t_2(m) \leq \sqrt[3]{t_1(m)}\ln t_1(m)$ for all $m \in N$;

(2) $\ln t_2(m) > \sqrt[3]{t_1(m)}\ln t_1(m)$ and $(\Delta', \ln t(m)) - \ln \tilde{\psi}(t(m)) \geq 0$ for all $m \in N$;

(3) $\ln t_2(m) > \sqrt[3]{t_1(m)}\ln t_1(m)$ and $(\Delta', \ln t(m)) - \ln \tilde{\psi}(t(m)) < 0$ for all $m \in N$.

In the case of the first and second possibilities, from relation (3) and inequality (5), we obtain the estimates

$$\beta(d) \geq \underline{\lim}_{m\to\infty} \frac{\ln \tilde{\psi}(t(m)) - (d, \ln t(m))}{\|\ln t(m)\|} \geq \lim_{m\to\infty}\left(-2^{-l_m}\right) = 0.$$

Let us consider the third possibility. Without loss of generality, for the sequence $\{t(m)\}$, we assume that either $t_1(m) \to \alpha$, $\alpha \in R$, as $m \to \infty$ or $t_1(m) \to \infty$ and $m \to \infty$ and

$$\left(\Delta' - d, \left(1, \sqrt[3]{t_1(m)}\right)\right) \geq 0$$

for all $m \in N$. If $t_1(m) \to \alpha$, $\alpha \in R$, as $m \to \infty$, then, by using the inequality $\Delta'_2 - d_2 \geq 0$ and relation (3), we obtain the estimates

$$\beta(d) \geq \varliminf_{m\to\infty} \frac{(\Delta_1' - d_1)\ln t_1(m) + (\Delta_2' - d_2)\ln t_2(m)}{\|\ln t(m)\|} \geq \varliminf_{m\to\infty} \frac{(\Delta_1' - d_1)\ln t_1(m)}{\|\ln t(m)\|} = 0;$$

otherwise, we have the estimate

$$\beta(d) \geq \varliminf_{m\to\infty} \frac{\left(\Delta' - d, \left(1, \sqrt[3]{t_1(m)}\,\right)\right)\ln t_1(m)}{\|\ln t(m)\|} \geq 0.$$

Therefore, if the curve D has the form (3) or (7), then the inclusion $D \subset \underline{D}_x$ is valid. Let us show that the lower exponent set $\underline{D}_x$ coincides with D. We choose an arbitrary point d of the lower exponent set $\underline{D}_x$ of the solution x. Then, by virtue of the relation $\ln\psi(t) = (\Delta', \ln t)$, we have the inequality $\Delta_2' - d_2 \geq 0$ in the direction $t = \left(t_1, t_1^{3\sqrt[3]{t_1}}\right) \in R^2_{>1}$, $t_1 \to \infty$. This inequality, together with the necessary properties of the lower exponent set, implies that $\underline{D}_x = D$.

In a similar way, one can show that the lower exponent set $\underline{D}_x$ of the solution x coincides with D if D has the form (5), (6), or (9).

2. In the case of the curve D of the form (9), consisting of an only point Δ of the plane R^2, we define the function ψ by the relation $\ln\psi(t) = (\Delta, \ln t)$, $t \in R^2_{>1}$. Obviously, in this case, the function x is a solution of the completely integrable equation (1_1) with infinitely differentiable bounded coefficients given by relation (4), its lower characteristic set P_x coincides with the lower characteristic set $P_\phi = \{p^0\}$ of the function ϕ, and $\underline{D}_x = \{\Delta\}$. The proof of Theorem 1 is complete.

The following assertion provides a complete description of the exponent sets of a solution $x \neq 0$ of system (1) with trivial characteristic sets.

Theorem 2. *The set D is the lower exponent set $\underline{D}_x$ (respectively, the upper exponent set $\bar{D}_x$) of some nontrivial solution x with trivial lower characteristic set P_x (respectively, trivial characteristic set Λ_x) of some completely integrable Pfaffian system* (1) *with bounded continuously differentiable coefficients if and only if this set either can be represented in the form of a continuous closed decreasing concave (respectively, convex) curve on a two-dimensional plane or is empty.*

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