Completion of Overdetermined Parabolic PDEs

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Abstract

In this paper we apply methods of commutative algebra to analysis of systems of PDEs. More precisely, we show that systems which are parabolic in a generalized sense are equivalent to certain completed systems which are parabolic in the standard sense. We also propose a constructive method to get this completion, and Gröbner basis methods, via symbol modules of the systems, play a central role in practical computations. Moreover, we can easily construct systems which are not parabolic in the generalized sense but nevertheless become parabolic when completed.

Key words: Overdetermined system, partial differential equation, parabolic system, completion, graded module, free resolution

1. Introduction

We continue here the work which was started in (Krupchyk et al., 2006). There we studied elliptic systems and their generalization due to Douglis and Nirenberg (1955). We showed that while Douglis–Nirenberg theory provides a framework to study certain square

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systems which are more general than standard elliptic systems, this theory is not needed when we allow overdetermined systems. Indeed, given any DN–elliptic system we can always complete it to an (overdetermined) elliptic system in the standard sense. Moreover, we constructed examples of systems which are not even DN–elliptic, but become elliptic when completed.

In the context of time dependent problems Solonnikov (1965) extended the approach of Douglis and Niremberg to parabolic problems. Again this generalization of parabolicity is convenient when one wants to restrict the attention to square systems. However, there are natural physical models which lead to overdetermined problems anyway, so in general we cannot restrict the attention to square systems only.

In this paper we show that the approach of (Krupchyk et al., 2006) can be adapted to parabolic problems. Evidently the theory becomes more complicated because we must take into account the special role played by the time variable. However, the conclusions can be stated in a similar way: any S-parabolic system becomes parabolic when completed, and there are systems which are not even S-parabolic initially, but become parabolic when completed. Our completion procedure is constructive; it just requires syzygy computations which can be done using Gröbner basis techniques, for example with the program SINGULAR [1].

There are several reasons why completion processes are important in the theory of PDEs. Many naturally occurring systems are either elliptic, parabolic or hyperbolic, and therefore there is an enormous literature dealing with these standard types of systems. Hence given some arbitrary system it is essential to determine in which class (if any) it belongs. So if a system is initially not parabolic but becomes parabolic when completed, then we can study the properties of solutions or well-posedness of boundary value problems by applying standard results to the completed system.

Of course, there are some results for square S-parabolic systems (Solonnikov, 1965; Eidelman, 1994), but as far as we know this theory has not been extended to overdetermined case. Anyway even in square case completing the system produces a system with better formal properties which helps in the further analysis of the system. Completion is even interesting from the point of view of numerical computations. For example in (Mohammadi and Tuomela, 2005) it was shown that in elliptic case the completed form of the system is solvable with simpler and more generic numerical methods than the original system. Indeed, complex numerical methods are often required to recover the properties which are missed by considering the initial system.

The structure of the paper is as follows. First we define S-parabolic systems and give some examples of situations where S-parabolic overdetermined systems naturally occur. Then we review and derive necessary results of commutative algebra needed to prove our results on parabolic systems. Next we discuss various completion procedures and their relevance. Finally, we prove our main results, and give examples which illustrate them.

[1] SINGULAR has been developed by the Singular team in Centre for Computer Algebra, University of Kaiserslautern under the direction of G.-M. Greuel, G. Pfister and H. Schönemann, see http://www.singular.uni-kl.de/

2. Parabolic systems

It is possible to study parabolic systems on smooth manifolds in a coordinate free way. However, our algebraic constructions require a choice of a coordinate system, hence the apparent generality of differential geometric language is not really useful here. Consequently we formulate all the theory in a fixed coordinate system.

2.1. Basic definitions

Let $\mathcal{X} \subset \mathbb{R}^n$ be open and $\mathcal{X}' = \mathcal{X} \times \mathbb{R}$. The coordinates of $\mathcal{X}'$ are denoted by $(x_1, \ldots, x_n, t)$. Let $A : C^\infty(\mathcal{X}', \mathbb{R}^m) \to C^\infty(\mathcal{X}', \mathbb{R}^k)$ be a differential operator. It is called an operator of order (q, b) (of order q with weight b with respect to t) if

$$Ay = \sum_{|\alpha|+br \le q} a_{\alpha,r}(x,t) \frac{\partial^{|\alpha|+r} y}{\partial x^\alpha \partial t^r} \tag{1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index and $a_{\alpha,r}$ are smooth $k \times m$ matrices.

Definition 1. The *full symbol* of A is a matrix $\sigma_F^b(A)$ given by

$$\sigma_F^b(A)(x,t,\xi,\tau) = \sum_{|\alpha|+br \le q} i^{|\alpha|+r} a_{\alpha,r}(x,t) \tau^r \xi^\alpha.$$

and the *principal b-homogeneous symbol* of A is a matrix $\sigma^b(A)$ defined by

$$\sigma^b(A)(x,t,\xi,\tau) = \sum_{|\alpha|+br = q} i^{|\alpha|+r} a_{\alpha,r}(x,t) \tau^r \xi^\alpha.$$

Let $\mathbb{C}^-$ be the closed lower half of the complex plane and set $\mathsf{G} = \{\mathbb{R}^n \times \mathbb{C}^-\} \setminus \{0\}$.

Definition 2. A differential operator A is *parabolic* if for any $(x,t) \in \mathcal{X}'$ and for all $(\xi,\tau) \in \mathsf{G}$ the maps $\sigma^b(A)(x,t,\xi,\tau) : \mathbb{C}^m \to \mathbb{C}^k$ are injective.

To generalise the notion of parabolicity Solonnikov (1965) introduced the concept of *weights* of the system, see also (Eidelman, 1994) for further discussion about these generalized parabolic systems. The weights are two sets of integers: we denote by s_i the weights for the equations, $1 \le i \le k$, and t_j the weights for the unknowns, $1 \le j \le m$. They must be chosen such that $s_i + t_j \ge q_{ij}$ where q_{ij} is the degree of polynomial $\left(\sigma_F^b(A)(x,t;\lambda\xi,\lambda^b\tau)\right)_{ij}$ in λ and if $s_i + t_j < 0$ then $\sigma_F^b(A)_{ij} \equiv 0$.

Definition 3. The *weighted (principal) symbol* of the operator A is a matrix $\sigma_w^b(A)$ whose entries are given by

$$\left(\sigma_w^b(A)(x,t,\xi,\tau)\right)_{i,j} = \sum_{|\alpha_{i,j}|+br_{i,j}=s_i+t_j} i^{|\alpha_{i,j}|+r_{i,j}} \left(a_{\alpha,r}(x,t)\right)_{i,j} \tau^{r_{i,j}} \xi^{\alpha_{i,j}}.$$

The operator A is *parabolic in the sense of Solonnikov* (S-parabolic) if for any $(x,t) \in \mathcal{X}'$ and for all $(\xi,\tau) \in \mathsf{G}$ the maps $\sigma_w^b(A)(x,t,\xi,\tau) : \mathbb{C}^m \to \mathbb{C}^k$ are injective.

It is clear that a system cannot be parabolic or S-parabolic if $k < m$. Hence in what follows we always suppose that $k \ge m$. Parabolic systems form a subclass of S-parabolic systems with weights

$$s_1 = \cdots = s_k = 0 \quad \text{and} \quad t_1 = \cdots = t_m = q$$

where q is the order of the operator (1). Obviously the weighted symbol $\sigma_w^b(A)$ remains unchanged if we replace all weights s_i by $s_i + c$ and all weights t_j by $t_j - c$ for some $c \in \mathbb{Z}$. The weighted principal symbol of A is called *reduced* if $t_1 = \cdots = t_m = \tilde{t}$ and s_i are arbitrary; in this case we denote the symbol by $\sigma_r^b(A)$.

An operator is S-parabolic if some choice of relevant weights exists and in general there are many different possible choices. However, it may in general not be easy to effectively find suitable weights. Note also that the property of being S-parabolic depends on the choice of coordinates for dependent variables.

2.2. Physical examples of overdetermined parabolic systems

Example 4. In some physical phenomena it is natural to combine several different interacting processes into a single model. In these cases the models do not in general fit into standard categories of elliptic, parabolic and hyperbolic systems. For example, the interaction of heat conduction with elastic waves in thermoelasticity leads to an operator which is "partly" parabolic and "partly" hyperbolic. So fixing our attention on one of the intervening processes leads to the consideration of overdetermined parabolic/hyperbolic systems.

In a simple (but still physically relevant) setting the *equations of thermoelasticity* can be written as follows (Dautray and Lions, 1987)

$$A(u,v) = \begin{cases} v_{tt} - \alpha\Delta v - \beta\nabla(\nabla\cdot v) + \gamma\nabla u = 0, \\ u_t - a\Delta u + b\nabla\cdot v_t = 0, \end{cases}$$

where v is the displacement field, u is the temperature and $\alpha, \beta, \gamma, a, b$ are positive constants. Then the operator A has the form $A(u,v) = A_1 u + A_2 v$ where

$$A_1 u = \begin{pmatrix} \gamma\nabla u \\ u_t - a\Delta u \end{pmatrix} \quad , \quad A_2 v = \begin{pmatrix} v_{tt} - \alpha\Delta v - \beta\nabla(\nabla\cdot v) \\ b\nabla\cdot v_t \end{pmatrix}.$$

Now A_1 (resp. A_2) is an overdetermined parabolic (resp. hyperbolic) operator. Since operators A_1 and A_2 can be classified in a standard way, there are many tools available to study them. This in turn is important in studying the properties of the original operator A.

Example 5. Let $\mathcal{X}$ be a domain in $\mathbb{R}^3$ with boundary $\partial\mathcal{X}$. Consider the initial boundary value problem of the *magnetohydrodynamic equations* (Davidson, 2001) in $\mathcal{X} \times (0, \infty)$

concerning the velocity field v, the magnetic field H and the scalar pressure p,

$$\begin{cases} v_t - \Delta v + v\nabla v + \nabla p + H \times \big(\nabla \times H\big) = 0 \\ H_t + \nabla \times \nabla \times H + v\nabla H - H\nabla v = 0 \qquad\qquad \text{in} \quad \mathcal{X} \times (0,\infty) \\ \nabla \cdot v = 0, \quad \nabla \cdot H = 0 \end{cases}$$
$$\begin{cases} v = 0 \\ \big(\nabla \times H\big) \times \nu = 0 \qquad \text{on} \quad \partial\mathcal{X} \times (0,\infty) \\ \langle \nu, H\rangle = 0 \end{cases} \tag{2}$$
$$\begin{cases} v(x,0) = a \\ H(x,0) = b \end{cases} \quad \text{on} \quad \mathcal{X}.$$

Here a and b are prescribed initial data and ν is the unit outward normal on $\partial\mathcal{X}$. The standard tool in the analysis of (2) is to study the following linear overdetermined problem:

$$Ay = \begin{cases} y_t + \nabla \times \nabla \times y = 0, \\ \nabla \cdot y = 0 \end{cases} \qquad \text{in} \quad \mathcal{X} \times (0,\infty), \tag{3}$$

with boundary conditions

$$By = \begin{cases} \big(\nabla \times y\big) \times \nu = 0 \\ \langle \nu, y\rangle = 0 \end{cases} \qquad \text{on} \quad \partial\mathcal{X} \times (0,\infty),$$

and initial condition $y(x,0) = b$.

Let us now show that the overdetermined system (3) is S-parabolic with $b = 2$. Indeed, the weighted principal symbol of A is

$$\sigma^2_w(A) = \begin{pmatrix} i\tau + \xi_2^2 + \xi_3^2 & -\xi_1\xi_2 & -\xi_1\xi_3 \\ -\xi_1\xi_2 & i\tau + \xi_1^2 + \xi_3^2 & -\xi_2\xi_3 \\ -\xi_1\xi_3 & -\xi_2\xi_3 & i\tau + \xi_1^2 + \xi_2^2 \\ i\xi_1 & i\xi_2 & i\xi_3 \end{pmatrix}$$

with weights $t_1 = t_2 = t_3 = 2$, $s_1 = s_2 = s_3 = 0$ and $s_4 = -1$. Computing all 3×3 minors of $\sigma^2_w(A)$, we get

$$i\xi_1(i\tau + |\xi|^2)^2, \; -i\xi_2(i\tau + |\xi|^2)^2, \; i\xi_3(i\tau + |\xi|^2)^2, \; i\tau(i\tau + |\xi|^2)^2$$

which shows our claim.

3. Free resolution of graded modules

In order to deal with S-parabolicity of differential operators one needs to study their weighted principal symbols. The structure of these symbols suggests that graded rings and modules are useful in analysing them. Here we introduce this necessary setting.

Let $\mathbb{A} = \mathbb{C}[\xi_1, \dots, \xi_n, \tau]$ be a polynomial ring in $n+1$ variables over the field of complex numbers. We introduce the weight $b > 1$ for the variable τ. Then the degree of monomial $\xi^\alpha\tau^r$ is $br + |\alpha|$ where α is a multi index. When considering homogeneous polynomials we always suppose that they are homogeneous with respect to this degree. Now we can consider $\mathbb{A}$ as a graded ring

$$\mathbb{A} = \mathbb{A}_0 \oplus \mathbb{A}_1 \oplus \dots$$

where $\mathbb{A}_s$ is the set of homogeneous polynomials of degree s. We also need a shifted graded ring $\mathbb{A}(\delta)$, $\delta \in \mathbb{Z}$, which is isomorphic to $\mathbb{A}$ as a ring and has a grading defined by

$$\mathbb{A}(\delta)_s = \mathbb{A}_{s+\delta}.$$

Then we define graded free modules to be

$$\mathbb{A}^k(d) = \mathbb{A}(d_1) \oplus \cdots \oplus \mathbb{A}(d_k),$$

for $d = (d_1, \dots, d_k) \in \mathbb{Z}^k$.

Let $S_0 : \mathbb{A}^{k_1}(-d^1) \to \mathbb{A}^{k_0}(-d^0)$ be a matrix which represents a module homomorphism of degree zero (Cox et al., 2005, p. 256). Then $M = \operatorname{coker}(S_0)$ is a finitely generated graded module. A *graded free resolution of* M is an exact complex

$$\cdots \longrightarrow \mathbb{A}^{k_2}(-d^2) \xrightarrow{S_1} \mathbb{A}^{k_1}(-d^1) \xrightarrow{S_0} \mathbb{A}^{k_0}(-d^0)$$

where $d^l \in \mathbb{Z}^{k_l}$ and S_l are matrices representing homomorphisms of degree zero. Recall that in a complex a composition of two consecutive maps is zero, and the exactness means that the image of each map is the kernel of the following map.

As S_l represents homomorphisms of degree zero, each of its ij entry is homogeneous of degree $d_j^{l+1} - d_i^l$ (or zero if $d_j^{l+1} - d_i^l < 0$). The image of S_l is the *lth syzygy module of* M and the matrix S_l is called the *lth syzygy matrix of* M.

We need the following theorem (Cox et al., 2005; Eisenbud, 1995).

Theorem 6. *Every finitely generated graded module has a finite graded free resolution of length at most* $n+1$.

Thus for M we obtain the exact complex

$$0 \longrightarrow \mathbb{A}^{k_r}(-d^r) \dots \longrightarrow \mathbb{A}^{k_2}(-d^2) \xrightarrow{S_1} \mathbb{A}^{k_1}(-d^1) \xrightarrow{S_0} \mathbb{A}^{k_0}(-d^0) \tag{4}$$

where $r \leq n+1$. Note that the free resolution is also useful in constructing compatibility complexes for overdetermined systems of PDEs (Krupchyk and Tuomela, 2007).

Let $I_j(S_0)$ denote the jth *Fitting ideal* of S_0 generated by all the $j \times j$-minors of S_0. It can be shown that the Fitting ideals depend only on $\operatorname{im}(S_0)$. The *rank* of S_0 in the sense of module theory, $\operatorname{rank}_{\mathbb{A}}(S_0)$, is the largest nonnegative integer r such that $I_r(S_0) \neq \langle 0 \rangle$. We put $I(S_0) = I_r(S_0)$.

Substituting some elements $(\bar{\xi}, \bar{\tau}) \in \mathbb{C}^{n+1}$ for the variables (ξ, τ) leads to a matrix $S_0(\bar{\xi}, \bar{\tau}) \in \mathbb{C}^{k_0 \times k_1}$. Its rank is denoted by $\operatorname{rank}\big(S_0(\bar{\xi}, \bar{\tau})\big)$. Obviously, we always have the inequality

$$\operatorname{rank}\big(S_0(\bar{\xi}, \bar{\tau})\big) \leq \operatorname{rank}_{\mathbb{A}}(S_0)$$

and for generic vectors $(\bar{\xi}, \bar{\tau})$ equality holds. Hence the specialization may affect the exactness of the sequence (4). The vectors leading to a smaller rank are defined by the zeros of $I(S_0)$, i.e., they correspond to the points of the variety $\mathsf{V}(I(S_0))$. Recall that the variety defined by the ideal $I(S_0)$ is the set of points in $\mathbb{C}^{n+1}$ on which all polynomials contained in $I(S_0)$ vanish.

From now on we use the notation (ξ, τ) for both the indeterminates of the polynomial ring $\mathbb{A}$ and vectors in $\mathbb{C}^{n+1}$. The intended meaning should be clear from the context.

Recall that the radical $\mathrm{rad}(I)$ of an ideal $I \subset \mathbb{A}$ consists of all polynomials f such that $f^n \in I$ for some $n \in \mathbb{N}$ and that $\mathsf{V}(I) = \mathsf{V}(\mathrm{rad}(I))$. Furthermore, if I, J are two ideals with $I \subset J$, then the corresponding varieties satisfy $\mathsf{V}(I) \supset \mathsf{V}(J)$. We need the following result (Eisenbud, 1995, p. 504).

Lemma 7. *The exactness of the complex (4) implies that*

$$\mathrm{rad}\big(I(S_0)\big) \subset \mathrm{rad}\big(I(S_1)\big).$$

Now it is clear that we obtain

$$\mathsf{V}\big(I(S_1)\big) = \mathsf{V}\big(\mathrm{rad}\big(I(S_1)\big)\big) \subset \mathsf{V}\big(\mathrm{rad}\big(I(S_0)\big)\big) = \mathsf{V}\big(I(S_0)\big). \tag{5}$$

Recall $\mathsf{G} = \big\{\mathbb{R}^n \times \mathbb{C}^-\big\} \setminus \{0\}$.

Lemma 8. *Suppose that we are given a graded free resolution* (4) *and*

$$\mathrm{rank}_{\mathbb{A}}(S_0) = \mathrm{rank}\big(S_0(\xi,\tau)\big) \qquad \forall(\xi,\tau) \in \mathsf{G}.$$

Then

$$\mathrm{rank}_{\mathbb{A}}(S_1) = \mathrm{rank}\big(S_1(\xi,\tau)\big) \qquad \forall(\xi,\tau) \in \mathsf{G}. \tag{6}$$

Proof. First note that $\mathrm{rank}\big(S_0(\xi,\tau)\big) < \mathrm{rank}_{\mathbb{A}}(S_0)$ is equivalent to $(\xi,\tau) \in \mathsf{V}\big(I(S_0)\big)$. Hence it follows from the hypothesis that $\mathsf{V}\big(I(S_0)\big) \subset \mathbb{C}^{n+1} \setminus \mathsf{G}$. But (5) implies that $\mathsf{V}\big(I(S_1)\big) \subset \mathbb{C}^{n+1} \setminus \mathsf{G}$ which yields (6). □

Lemma 9. *Under the assumptions of Lemma 8, the complex*

$$\mathbb{C}^{k_2} \xrightarrow{S_1(\xi,\tau)} \mathbb{C}^{k_1} \xrightarrow{S_0(\xi,\tau)} \mathbb{C}^{k_0} \tag{7}$$

is exact for all vectors $(\xi,\tau) \in \mathsf{G}$.

Proof. Since (4) is exact, we have (Eisenbud, 1995, p. 500)

$$k_1 = \mathrm{rank}(\mathbb{A}^{k_1}(-d^1)) = \mathrm{rank}_{\mathbb{A}}(S_0) + \mathrm{rank}_{\mathbb{A}}(S_1).$$

Using Lemma 8, we get

$$\begin{aligned} k_1 =& \mathrm{rank}\big(S_0(\xi,\tau)\big) + \mathrm{rank}\big(S_1(\xi,\tau)\big) \\ =& \dim\mathrm{im}\big(S_0(\xi,\tau)\big) + \dim\mathrm{im}\big(S_1(\xi,\tau)\big) \quad \forall(\xi,\tau) \in \mathsf{G} \end{aligned} \tag{8}$$

Since $S_0 S_1 = 0$, we always have

$$\mathrm{im}\big(S_1(\xi,\tau)\big) \subset \ker\big(S_0(\xi,\tau)\big) \qquad \forall(\xi,\tau) \in \mathbb{C}^{n+1}.$$

Furthermore, $S_0(\xi,\tau)$ satisfies trivially $\dim\mathrm{im}\big(S_0(\xi,\tau)\big) = k_1 - \dim\ker\big(S_0(\xi,\tau)\big)$. So (8) implies

$$\dim\ker\big(S_0(\xi,\tau)\big) = \dim\mathrm{im}\big(S_1(\xi,\tau)\big) \qquad \forall(\xi,\tau) \in \mathsf{G}$$

Together with the inclusion above, this observation entails

$$\mathrm{im}\big(S_1(\xi,\tau)\big) = \ker\big(S_0(\xi,\tau)\big) \qquad \forall(\xi,\tau) \in \mathsf{G}$$

and hence the exactness of (7). □

If we apply the functor $\mathrm{Hom}_{\mathbb{C}}(\cdot, \mathbb{C})$ to an exact sequence of vector spaces i.e. if we dualise the sequence, then by a standard result in homological algebra we obtain again an exact sequence. At the level of matrices this yields the following corollary to the above lemma.

Corollary 10. *Under the assumptions of Lemma 8, the transposed complex*

$$\mathbb{C}^{k_0} \xrightarrow{S_0^T(\xi,\tau)} \mathbb{C}^{k_1} \xrightarrow{S_1^T(\xi,\tau)} \mathbb{C}^{k_2}$$

is also exact for all $(\xi, \tau) \in \mathsf{G}$.

However, the exactness of the complex (4) of free modules over a ring $\mathbb{A}$ does not imply the exactness of the dual complex, as the functor $\mathrm{Hom}_{\mathbb{A}}(\cdot, \mathbb{A})$ is in general only left exact (Eisenbud, 1995).

Now let $C^T : \mathbb{A}^{k_1}(-d^1) \to \mathbb{A}^{k_0}(-d^0)$ be a matrix which represents a module homomorphism of degree zero. Then by Theorem 6 there is a graded free resolution of $\mathrm{coker}(C^T)$ as in diagram (4) with $S_0 = C^T$. We suppose that $k_1 > k_0$; this implies that the first syzygy matrix S_1 of $\mathrm{coker}(C^T)$ is nonzero.

Let us now consider

$$B = \big(C, B_1\big) \quad \text{and} \quad B' = \begin{pmatrix} C & 0 \\ B_2 & S_1^T B_1 \end{pmatrix}$$

where B_1 and B_2 are arbitrary matrices of appropriate sizes. To prove our main result we need the following technical lemma.

Lemma 11. *Under the above assumptions, we have*

$$\begin{aligned} \ker\big(B(\xi,\tau)\big) = \{0\} \quad &\forall\ (\xi,\tau) \in \mathsf{G} \\ &\Downarrow \\ \ker\big(B'(\xi,\tau)\big) = \{0\} \quad &\forall\ (\xi,\tau) \in \mathsf{G} \end{aligned}$$

Proof. Suppose that there is a vector $(\hat{\xi}, \hat{\tau}) \in \mathsf{G}$ such that $\ker\big(B'(\hat{\xi}, \hat{\tau})\big) \neq \{0\}$. Then $B'(\hat{\xi}, \hat{\tau})v = 0$ for some $v = \big(\tilde{v}, \hat{v}\big) \neq 0$. This implies that $C(\hat{\xi}, \hat{\tau})\tilde{v} = 0$. Since $\ker\big(B(\hat{\xi}, \hat{\tau})\big) = \{0\}$, we have $\ker\big(C(\hat{\xi}, \hat{\tau})\big) = \{0\}$ and then it follows that $\tilde{v} = 0$. Thus we get

$$S_1^T(\hat{\xi}, \hat{\tau}) B_1(\hat{\xi}, \hat{\tau})\hat{v} = 0 \quad \text{and} \quad B_1(\hat{\xi}, \hat{\tau})\hat{v} \in \ker\big(S_1^T(\hat{\xi}, \hat{\tau})\big).$$

The fact that $\ker\big(C(\hat{\xi}, \hat{\tau})\big) = \{0\}$ yields that $\mathrm{rank}_{\mathbb{A}}(C^T) = \mathrm{rank}\big(C^T(\hat{\xi}, \hat{\tau})\big)$. Hence Corollary 10 implies that $\ker\big(S_1^T(\hat{\xi}, \hat{\tau})\big) = \mathrm{im}\big(C(\hat{\xi}, \hat{\tau})\big)$. So there is some $\hat{u}$ such that $C(\hat{\xi}, \hat{\tau})\hat{u} + B_1(\hat{\xi}, \hat{\tau})\hat{v} = 0$. Putting $u = \big(\hat{u}, \hat{v}\big) \neq 0$ implies that $B(\hat{\xi}, \hat{\tau})u = 0$. But this contradicts our assumption that $\ker\big(B(\xi, \tau)\big) = \{0\}$ for all $(\xi, \tau) \in \mathsf{G}$. □

4. Why *b* must be even?

In terms of varieties we can express S-parabolicity as follows: a differential operator A in (1) is S-parabolic if and only if

$$\mathsf{V}\big(I_m(\sigma_w^b(A))\big) \subset \mathbb{C}^{n+1} \setminus \mathsf{G}.$$

As in elliptic case, we may call $\mathsf{V}\big(I_m(\sigma_w^b(A)\big)$ the *characteristic variety* of A. Now recall that A is an operator of *finite type* (Pommaret, 1978; Krupchyk and Tuomela, 2006; Seiler, 2001), if

$$\mathsf{V}\big(I_m(\sigma_w^b(A))\big) = \{0\}\ .$$

As an example of a system of finite type, one can consider the following system

$$\begin{cases} y_t + y_{xxx} = 0, \\ y_t + 2y_{xxx} = 0. \end{cases}$$

Intuitively finite type means that there are only finite number of degrees of freedom in the system, or in other words that the formal solution space is finite dimensional.

Note that the above system is also parabolic with $b = 3$. More generally we have

Lemma 12. *If A is S-parabolic and b is odd, then A is of finite type.*

Proof. Let $b = 2\tilde{b} + 1$. By the definition of the weighted principal symbol, each $m \times m$ minor of $\sigma_w^b(A)$ is a homogeneous polynomial in (ξ, τ) of the following form

$$\sum_{|\alpha|+br=\tilde{q}} i^{|\alpha|+r} c_{\alpha,r}(x,t)\tau^r\xi^\alpha = i^{\tilde{q}} \sum_{|\alpha|+(2\tilde{b}+1)r=\tilde{q}} (-1)^{\tilde{b}r} c_{\alpha,r}(x,t)\tau^r\xi^\alpha$$

with some real valued functions $c_{\alpha,r}(x,t)$. Thus, for any $(x,t) \in \mathcal{X}'$ and $0 \neq \xi \in \mathbb{R}^n$, each minor considered as a polynomial in τ has real roots or pairs of complex conjugate roots. This and the fact that A is S-parabolic, which is equivalent to $\mathsf{V}\big(I_m(\sigma_w^b(A))\big) \subset \mathbb{C}^{n+1} \setminus \mathsf{G}$, imply that $\mathsf{V}\big(I_m(\sigma_w^b(A))\big) = \{0\}$. □

This result shows that we may as well always suppose that b is even; indeed systems of finite type are also elliptic, so no special parabolic theory is needed for them.

5. Completion to parabolic system

5.1. Different approaches to completion of differential systems

There are (at least) three different frameworks to study general systems of PDEs: formal theory (Spencer, 1969; Pommaret, 1978; Seiler, 2001), differential algrebra (Ritt, 1966; Kolchin, 1973) and exterior differential systems (Bryant et al., 1991). In some sense all these theories provide a way to produce a "complete" system of PDEs which is more easily analysed than the original system. Note that all these theories have their origins in classical works of Riquier (1910), Janet (1929) and É. Cartan (1945).

The construction of completed system does not change the formal solution space of the system, hence one can say that the completion process produces a system which is equivalent to the original one. One can make the notion of equivalence more rigorous using the language of homological algebra (Tarkhanov, 1995; Krupchyk and Tuomela, 2006).

In all approaches there are algorithms for actually constructing the completed or involutive form of the system. We do not consider these in any detail and refer to (Mansfield, 2001; Seiler, 2001) for further discussion and references about various aspects of constructing the completed form.

Anyway we claim that the formal theory is the most appropriate approach for our purposes; this is because ultimately we would like to analyse boundary value problems, and this is difficult or impossible using the other two approaches. In differential algebraic context it is not even clear how to define the boundary. In case of exterior differential systems we have the necessary geometry to set up boundary conditions; however, the notion of symbol of PDE is missing or at least hidden in this formalism, and hence the study of well posedness of the problem is more complicated. Of course there are situations where the problem is naturally stated as an exterior differential system and if this is the case then obviously it is reasonable that the further analysis of the problem is conducted in that framework. However, the problems we have in mind appear as systems of PDEs and translating them to exterior differential systems, while possible, does not seem to offer any advantages.

So below we only mention some aspects related to formal theory and do not discuss the other two approaches any further. Now one of the main results of the formal theory says that *any* system of PDEs can be transformed in a finite number of steps to *involutive* form (Spencer, 1969; Pommaret, 1978; Seiler, 2001). The reason to compute the involutive form is that it contains "all" necessary information about the system, and hence this form is better suited to the further and more precise analysis of the system. For example we showed in (Krupchyk et al., 2006) that it is in general impossible to know if the system is elliptic if one does not first transform it to the involutive form.

But this geometric framework is mainly adapted to treat elliptic problems: all coordinates are in some sense equivalent or in more concrete terms can be interpreted as "space" coordinates. Evidently we can compute the involutive form of any time dependent problem by simply ignoring the role of time variable. But the resulting system is not in general useful for any further analysis of the system. Hence in time dependent problems, the naive application of this theory fails because it is in the nature of the problem that the time variable or coordinate has a special status which must be incorporated into theory. In spirit this is contrary to geometric way of thinking where one tries to formulate everything in a coordinate free way. But it is obvious that in the analysis of the time dependent PDEs the time variable plays a special role which cannot be ignored. At the very least this becomes clear when considering initial-boundary value problems: the domain of the problem is geometrically a cylinder and on the bottom of the cylinder one needs initial conditions and on the side one needs boundary conditions, and the nature of these two types of conditions is totally different.

In (Dudnikov and Samborski, 1996; Feldman, 1987) one tries to circumvent this difficulty by introducing anisotropic jet spaces; this means that we build the time variable in the theory from the very beginning. In this framework we can try to imitate the formal theory and try to prove the basic theorems in this new setting. So in (Dudnikov and Samborski, 1996) one finds a definition of *formal integrability* adapted to this context. However, apparently nobody has developed the theory any further. In particular, to apply the theory in a constructive way the notion of involutive symbol should be extended to the parabolic case.

In what follows we do not attempt to develop the formal theory in the time dependent case. Instead we propose an independent completion procedure which produces a parabolic system from any S-parabolic one. This is not a completion to involutive form but it yields a system with better formal properties than the original one. Our completion procedure is constructive and based on syzygy computations in graded modules.

5.2. Completion procedure

Our goal in this section is to show that if weights exist such that the linear differential operator A is S-parabolic, then there is a completion of A which can be constructed in a finite number of steps and which leads to an equivalent operator that is parabolic without weights. Thus we may dispense with the introduction of weights, if we can always complete the system before the classification.

Lemma 13. *Let A be an S-parabolic operator with the reduced weighted principal symbol. Then we can always get an equivalent parabolic operator by differentiating some equations and adding them to the system.*

Proof. We can always suppose that for the reduced symbol $\sigma_r^b(A)$, we have the weights $t_1 = \cdots = t_m = 0$. Assume that a weight $s_i > 0$ is not divisible by b. Then none of the entries of the ith row of the reduced symbol $\sigma_r^b(A)$ contain monomials only in a variable τ. Since these entries are homogeneous polynomial of order s_i in (ξ, τ), for $\xi = 0$ the ith row of $\sigma_r^b(A)$ is zero. Recall that when considering homogeneous polynomials we always suppose that they are homogeneous with respect to the degree defined in Section 3. Thus, the ith row does not influence the injectivity of $\sigma_r^b(A)(0, \tau)$ with $0 \neq \tau \in \mathbb{C}^-$.

Let now $\tilde{s}_i$ be the minimal integer that is bigger than s_i and divisible by b. Then by the above observation, adding to the system all equations obtained by differentiating the ith equation $\tilde{s}_i - s_i$ times with respect to each variable x_ℓ, $\ell = 1, \dots, n$, we get an S-parabolic system. Indeed, choosing the weight $\tilde{s}_i$ for the ith equation as well as for each of new equations, and keeping all other weights as before we get an injective reduced symbol in G. Hence, one can always complete a system to an equivalent system which is S-parabolic with the reduced symbol where all nonzero weights s_i are divisible by b.

Now set $\hat{s} = \max s_i > 0$. If the row corresponding to the ith equation of the reduced symbol is nonzero, we differentiate this equation $(\hat{s} - s_i)/b$-times with respect to t and $\hat{s} - s_i$ times with respect to each variable x_ℓ, $\ell = 1, \dots, n$, and add these differential consequences to the system. It is obvious that the resulting system is parabolic. □

Theorem 14. *Any S-parabolic operator can be completed to an equivalent parabolic operator.*

Proof. Let us consider an S-parabolic operator A given in (1). Then we can always suppose that its weighted principal symbol is decomposed as

$$\sigma_w^b A = (\sigma_r^b A_1, \dots, \sigma_r^b A_\beta)$$

where the weights for dependent variables are arranged as follows

$$t_1 = \cdots = t_{j_1} > t_{j_1+1} = \cdots = t_{j_1+j_2} > \cdots > t_{j_1+\cdots+j_{\beta-1}+1} = \cdots = t_m, \\ m = j_1 + \cdots + j_\beta \ .$$

Here the block $\sigma_r^b A_l$ is a $k \times j_l$ matrix whose entries are homogeneous polynomials in (ξ, τ) of degree $s_i + t_{j_1+\cdots+j_l}$.

Now $(\sigma_r^b A_1)^T$ represents a module homomorphism of degree zero,

$$(\sigma_r^b A_1)^T \, : \, \mathbb{A}^k(-d^1) \to \mathbb{A}^{j_1}(-d^0)$$

with $d_i^1 = s_i + t_1$, $i = 1, \ldots, k$, and $d^0 = 0$. By Theorem 6 there is a finite graded free resolution of $\operatorname{coker}\big((\sigma_r^b A_1)^T\big)$. Since $j_1 < k$, the first syzygy matrix S_1 in this resolution is nonzero and represents a module homomorphism of degree zero,

$$S_1 : \mathbb{A}^{k_2}(-d^2) \to \mathbb{A}^k(-d^1)$$

with some $d^2 \in \mathbb{Z}^{k_2}$. Denoting by v^r, $r = 1, \ldots, k_2$, the columns of S_1, we have that v_i^r is a homogeneous polynomial of degree $d_r^2 - s_i - t_1$ (or zero if $d_r^2 - s_i - t_1 < 0$).

Substituting $i^{-1}\partial/\partial x_j$ and $i^{-1}\partial/\partial_t$ (where i is the imaginary unit) for ξ_j and τ, respectively, in the matrix S_1, we construct the differential operator $\hat{S}_1$. Let us now consider the operator $A^{(1)} = (A, \hat{S}_1^T A)$. If we choose $t_j^{(1)} = t_j + 1$ for $j > j_1$, $s_{k+r}^{(1)} = d_r^2 - t_{j_1} - 1$, $r = 1, \ldots, k_2$, and all other weights as in $\sigma_w A$, then it is easily seen that the weighted principal symbol of $A^{(1)}$ is of the form

$$\sigma_w A^{(1)} = \begin{pmatrix} \sigma_r A_1 & 0 \\ B & S_1^T\big(\sigma_r A_2, \ldots, \sigma_r A_\beta\big) \end{pmatrix}$$

with some matrix B of appropriate size.

Since the symbol $\sigma_w^b A$ is injective in G, Lemma 11 implies that the symbol $\sigma_w^b A^{(1)}$ is also injective in G. So we can apply the same arguments to the operator $A^{(1)}$ and proceed in this way until we obtain an operator $A^{(\nu)}$ such that $t_{j_1} = t_{j_1+j_2}^{(\nu)}$. Thus in a finite number of steps we have reduced an S-parabolic operator with β block columns to an equivalent operator with $\beta - 1$ block columns.

Hence after a finite number of steps we obtain an operator which is equivalent to the original operator and which has a parabolic reduced symbol. But according to Lemma 13, this suffices to prove our claim. □

Thus, the notion of S-parabolicity does not define a larger class of systems than parabolic ones. Moreover, the proof of Theorem 14 gives us more than an existence result. It contains a *completion algorithm* which produces an equivalent parabolic system from any S-parabolic one. In the following example where we illustrate our algorithm we have used Singular to compute the relevant syzygies.

Example 15. Consider the following system

$$Ay = \begin{cases} y^1_{1,00} - y^1_{0,20} + y^1_{0,10} + y^2 = 0, \\ y^1_{1,00} - y^1_{0,02} - y^2 = 0 \end{cases}$$

where we have used the notation

$$y_{r,\alpha} = \frac{\partial^{|\alpha|+r} y}{\partial x^\alpha \partial t^r}.$$

Hence $b = 2$ and the weighted principal symbol of this system is

$$\sigma_w^2(A) = \begin{pmatrix} i\tau + \xi_1^2 & 1 \\ i\tau + \xi_2^2 & -1 \end{pmatrix}$$

with weights $t_1 = 0$, $t_2 = -2$ and $s_1 = s_2 = 2$. The operator A is S-parabolic because $\det \sigma_w^2(A) = -2i\tau - |\xi|^2 \neq 0$ for $(\xi, \tau) \in \mathsf{G}$.

We write the system and its weighted principal symbol as $Ay = A_1 y^1 + A_2 y^2$ and $\sigma_w^2(A) = (\sigma_r^2(A_1), \sigma_r^2(A_2))$. Computing the first syzygy matrix of the module $\operatorname{coker}\big(\sigma_r^2(A_1)\big)^T$, we get

$$S_1 = \begin{pmatrix} -i\tau - \xi_2^2 \\ i\tau + \xi_1^2 \end{pmatrix}.$$

Thus, in the notation of Theorem 14, we have $d^2 = 4$. The differential operator corresponding to S_1 is given by

$$\hat{S}_1 = \begin{pmatrix} -\partial_{1,00} + \partial_{0,02} \\ \partial_{1,00} - \partial_{0,20} \end{pmatrix}$$

where

$$\partial_{r,\alpha} = \frac{\partial^{|\alpha|+r}}{\partial x^\alpha \partial t^r}$$

Hence we obtain

$$\hat{S}_1^T A y = -y^1_{1,10} + y^1_{0,12} - 2y^2_{1,00} + y^2_{0,20} + y^2_{0,02}\ .$$

The weighted principal symbol of the operator $A^{(1)} = \big(A, \hat{S}_1^T A\big)$ is

$$\sigma_w^2(A^{(1)}) = \begin{pmatrix} i\tau + \xi_1^2 & 0 \\ i\tau + \xi_2^2 & 0 \\ -i\xi_1(i\tau + \xi_2^2) & -2i\tau - \xi_1^2 - \xi_2^2 \end{pmatrix}$$

with the weights $t_2^{(1)} = -1$, $s_3^{(1)} = d^2 - 1 = 3$ and all other weights are as in $\sigma_w^2(A)$. Now $t_1^{(1)} - t_2^{(1)} = 1$ and the first syzygy matrix of $\operatorname{coker}\big(\sigma_r^2(A_1^{(1)})\big)^T$ is

$$S_1^{(1)} = \begin{pmatrix} 0 & -\xi_2^2 - i\tau \\ i\xi_1 & i\tau \\ 1 & i\xi_1 \end{pmatrix}.$$

Now the shifts are $d_1^{(2)} = 3$ and $d_2^{(2)} = 4$, and the corresponding differential operator is

$$\hat{S}_1^{(1)} = \begin{pmatrix} 0 & \partial_{0,02} - \partial_{1,00} \\ \partial_{0,10} & \partial_{1,00} \\ 1 & \partial_{0,10} \end{pmatrix}.$$

Thus,

$$\big(\hat{S}_1^{(1)}\big)^T A^{(1)} y = \begin{cases} -2y^2_{1,00} + y^2_{0,20} + y^2_{0,02} - y^2_{0,10} = 0 \\ y^1_{0,12} - y^1_{1,10} - 2y^2_{1,10} + y^2_{0,30} + y^2_{0,12} + y^2_{0,02} - 2y^2_{1,00} = 0 \end{cases}$$

Hence the weighted principal symbol of the operator $A^{(2)} = \big(A^{(1)}, (\hat{S}_1^{(1)})^T A^{(1)}\big)$ is

$$\sigma_r^2(A^{(2)}) = \begin{pmatrix} i\tau + \xi_1^2 & 0 \\ i\tau + \xi_2^2 & 0 \\ -i\xi_1(i\tau + \xi_2^2) & 0 \\ 0 & -2i\tau - \xi_1^2 - \xi_2^2 \\ -i\xi_1(i\tau + \xi_2^2) & i\xi_1(-2i\tau - \xi_1^2 - \xi_2^2) \end{pmatrix}$$

with the weights $t_2^{(2)} = 0$, $s_4^{(2)} = d_1^{(2)} - 1 = 2$, $s_5^{(2)} = d_2^{(2)} - 1 = 3$ and all other weights are as in $\sigma_w^2(A^{(1)})$. So the operator $A^{(2)}$ is S-parabolic with respect to the reduced symbol. Then proceeding as in Lemma 13 we finally obtain a parabolic system.

The following example shows that it may not be possible to find weights such that the original system is S-parabolic, although the system becomes parabolic after the completion. Hence we conclude that the weights are neither necessary nor sufficient for deciding parabolicity of a differential operator.

Example 16. Let us consider the following system

$$Ay = \begin{cases} y^1_{2,0} - 2y^1_{1,2} + y^1_{0,4} + y^2_{2,0} - 2y^2_{1,2} + y^2_{0,4} + y^2_{0,2} - y^3_{1,2} + y^3_{0,4} = 0, \\ -y^1_{1,2} + y^1_{0,4} - y^2_{1,2} + y^2_{0,4} + y^2_{1,0} + y^3_{0,4} = 0, \\ -y^1_{0,2} + y^2_{1,0} - y^2_{0,2} - y^3_{1,0} + y^3_{0,2} = 0, \end{cases}$$

of order 4 with $b = 2$. If we choose weights $t_1 = t_2 = t_3 = 4$, $s_1 = s_2 = 0$ and $s_3 = -2$, we get the following reduced principal symbol

$$\sigma_r^2(A) = \begin{pmatrix} (i\tau + \xi^2)^2 & (i\tau + \xi^2)^2 & (i\tau + \xi^2)\xi^2 \\ (i\tau + \xi^2)\xi^2 & (i\tau + \xi^2)\xi^2 & \xi^4 \\ \xi^2 & i\tau + \xi^2 & -(i\tau + \xi^2) \end{pmatrix}$$

which is clearly not parabolic because the first and the second rows are linearly dependent. Some relevant information is contained in the second-order terms $y^2_{0,2}$ and $y^2_{1,0}$ in the first and second equations, respectively. As the fourth-order derivatives of y^2 are present in these equations, it is not possible to choose weights such that these terms enter the symbol and, therefore, the system cannot be S-parabolic.

Applying the operators $\partial_{0,2}$ and $\partial_{1,0} - \partial_{0,2}$ to the first and the second equations, respectively, and adding these equations, we get an integrability condition

$$y^2_{2,0} - y^2_{1,2} + y^2_{0,4} = 0.$$

Adding the integrability condition to the system gives

$$A^{(1)}y = \begin{cases} y^1_{2,0} - 2y^1_{1,2} + y^1_{0,4} + y^2_{2,0} - 2y^2_{1,2} + y^2_{0,4} + y^2_{0,2} - y^3_{1,2} + y^3_{0,4} = 0, \\ -y^1_{1,2} + y^1_{0,4} - y^2_{1,2} + y^2_{0,4} + y^2_{1,0} + y^3_{0,4} = 0, \\ -y^1_{0,2} + y^2_{1,0} - y^2_{0,2} - y^3_{1,0} + y^3_{0,2} = 0, \\ y^2_{2,0} - y^2_{1,2} + y^2_{0,4} = 0. \end{cases}$$

The reduced principal symbol of $A^{(1)}$ is

$$\sigma_r^2(A^{(1)}) = \begin{pmatrix} (i\tau+\xi^2)^2 & (i\tau+\xi^2)^2 & (i\tau+\xi^2)\xi^2 \\ (i\tau+\xi^2)\xi^2 & (i\tau+\xi^2)\xi^2 & \xi^4 \\ \xi^2 & i\tau+\xi^2 & -(i\tau+\xi^2) \\ 0 & -\tau^2+i\tau\xi^2+\xi^4 & 0 \end{pmatrix}$$

with weights $t_1 = t_2 = t_3 = 4$, $s_1 = s_2 = s_4 = 0$ and $s_3 = -2$. Since the determinant consisting of the first, third and fourth rows of the above symbol is equal to

$$-i(\tau - \frac{1}{2}(i-\sqrt{3})\xi^2)(\tau - \frac{1}{2}(i+\sqrt{3})\xi^2)(\tau - i\xi^2)(\tau-(-1+i)\xi^2)(\tau-(1+i)\xi^2),$$

the reduced symbol $\sigma_r^2(A^{(1)})$ is injective in G. Thus simply applying the operator $\partial_{1,0} - \partial_{0,2}$ to the third equation produces a parabolic system.

Using the idea of this example it is quite easy to construct other examples which are not S-parabolic, but become parabolic when completed. In the above example we found the integrability condition by first diffentiating some equations, and then taking appropriate combinations to eliminate highest order derivatives. In geometric terms we first prolonged our system to a higher order jet space and then projected the resulting system back to the initial jet space. So the computations in the example were not a special trick but in this way one can systematically look for integrability conditions in general. Adding these integrability conditions may then lead to a parabolic system.

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