# Bounds for the group velocity of electromagnetic signals in two phase materials

K. Sølna*  G. W. Milton†

July 11, 1999

**Abstract**

The energy associated with an electromagnetic signal travels in general with the group velocity. We consider the group velocity of a *composite* and ask the question whether the group velocity in the composite can be higher than in either of the constituent phases. Here we show that it can, and by a large factor.

The key point is that the group velocity depends on both the refractive index and the dispersion. By combining one phase with high refractive index and low dispersion with another phase with low refractive index and high dispersion, the composite can be made to exhibit comparatively low refractive index and low dispersion and hence a large group velocity. In particular this can be realized when the dispersion relations of both phases are described by a Lorentzian model and one phase is close to resonance.

The 'speed-up' is largest in a laminate microgeometry, but can be made large also in isotropic microstructures, described by a Maxwell-Garnett model. These geometries attain the bounds on the speed-up that we derive. The group velocity can also be smaller in the composite than in the phases and we derive bounds for the possible 'slow-down'. These bounds are attained by similar geometries as those that realize the optimal bounds for the speed-up.

*Department of Mathematics, University of Utah, Salt Lake City, Utah 84112; solna@math.utah.edu;
†Department of Mathematics, University of Utah, Salt lake City UT 84112; milton@math.utah.edu;
Fax: (801)5814148

**Keywords**: Electromagnetic wave propagation, group velocity, homogenization, effective parameters.

## 1 Introduction

For electromagnetic wave propagation a lot is known about the effective properties of materials in the time harmonic case [4, 7]. Expressions for the effective dielectric tensor $\varepsilon_*(\omega)$ often give an accurate description of a materials response to a field that varies at a constant frequency, $\omega$. Not so much is known about propagation of pulses. Our objective is to characterize and compute bounds for the velocity of wave-pulses that propagates in composites.

We consider a dielectric composite of two isotropic phases that is subject to some time dependent field $\mathbf{E}(t)$. We work in the quasistatic regime, that is we assume that the time variation of the electric field is sufficiently slow such that the curl of the electric field varies on a length scale large relative to the microstructure of the material. An important aspect of the propagating pulse is its group velocity, in general this corresponds to the velocity with which energy travels. An interesting question is: can one derive a bound for the group velocity of the composite when this is measured relative to the group velocities of the two phases? In this paper we present such bounds.

Let $k$ be the wave number and $\omega$ the (temporal) frequency. For light waves in a medium with magnetic permeability $\mu \equiv 1$ the relation between $\omega$ and $k$ is given by

$$\omega(k) \;=\; \frac{ck}{\sqrt{\epsilon^*(k)}} \tag{1}$$

with $c$ being the velocity of light in vacuum and $\epsilon^*$ the (effective) dielectric constant of the medium. If the initial wave packet is (narrowly) centered at $k = k_0$, corresponding to $\omega = \omega_0$, it propagates with the group velocity

$$v_g(k_0) \;=\; [dk/d\omega(\omega_0)]^{-1} \;=\; d\omega/dk(k_0), \tag{2}$$

see [5]. Note that if $\epsilon^*(k) \equiv \epsilon^*$, then $v_g = v_p = \omega(k_0)/k_0 = c/\sqrt{\epsilon^*}$ with $v_p$ the phase velocity. In general $v_g \neq v_p$ and the wave propagation is dispersive.

## 2 Bounds for the group velocity

The composite is constructed from two isotropic dielectric phases. The parameters

$$\epsilon_i(\omega_0) \quad \text{and} \quad d\epsilon_i/d\omega(\omega_0) \tag{3}$$

characterize material $i = 1$ or 2 with $\omega_0$ being the frequency at which we evaluate the group velocity. The phases are labeled so that $\epsilon_1(\omega_0) \geq \epsilon_2(\omega_0)$ and we let $p$ denote the volume fraction of material one. We assume

- The quasistatic regime.
- That the magnetic permeability is constant: $\mu \equiv 1$.
- That the dielectric constant is *real.*

Thus, we do not consider dissipation and the relationship between $\omega$ and $k$ is given by (1).

Next, we present bounds for the 'speed-up' and 'slow-down' of the composite relative to the constituents. Here we present bounds in the general anisotropic case, see [10] for bounds if the composite is constrained to be isotropic or if the volume fraction is fixed.

The bounds for the group velocity are obtained through joint bounds for the *effective* values of the parameters (3) in the *composite* [10], which derive from joint bounds on $\epsilon^*$ and $d\epsilon^*/d\epsilon_1$ derived in [1, 2, 3, 8, 9, 11].

### 2.1 Maximal increase in group velocity

Let $\{v_1, v_2, v^*\}$ be respectively the group velocities of the phases and the composite and define

$$G = \min[v^*/v_1, v^*/v_2]. \tag{4}$$

We want to maximize this quantity with respect to the *geometry* of the composite and the *volume fraction.* For electromagnetic wave propagation in the quasistatic regime we find that the sharp bound for $G$ is

$$G \leq \mathcal{G}(h,\beta) = \begin{cases} \frac{(1-h)\beta}{2\sqrt{h(1-\beta)(\beta-h)}} & \text{for } \frac{2h}{1+h} \leq \beta \leq \sqrt{h} \\ \frac{1-h}{2\sqrt{(1-\beta)(\beta-h)}} & \text{for } \sqrt{h} \leq \beta \leq \frac{1+h}{2} \\ 1 & \text{else} \end{cases} \tag{5}$$

with

$$
\begin{aligned}
h &= \epsilon_2(\omega_0)/\epsilon_1(\omega_0) \ \leq \ 1 \\
\beta &= \left[\frac{\omega d\epsilon_1/d\omega/(2\epsilon_1)+1}{\omega d\epsilon_2/d\omega/(2\epsilon_2)+1}\right]_{\omega_0}.
\end{aligned} \tag{6}
$$

The analysis we present in [10] gives bounds also for $v^*/v_1$ and $v^*/v_2$. The bound (5) is realized by waves propagating in a laminated material, with the layers orthogonal to the direction of propagation and with volume fraction of phase one

$$
p = \frac{h(1-2\beta+h)}{(1-h)(\beta-h)}. \tag{7}
$$

Moreover, we find that

$$
\max_{\beta} \mathcal{G}(h,\beta) = \mathcal{G}(h,\sqrt{h}) = \frac{1+\sqrt{h}}{2h^{1/4}} \sim \frac{1}{2h^{1/4}} \qquad as \ \ h \downarrow 0. \tag{8}
$$

At this maximal speed-up $p = \sqrt{h}/(1+\sqrt{h})$, $v_1 = v_2$ and the effective dielectric constant of the laminate is $\epsilon^* = \sqrt{h}\epsilon_1$.

Thus, by choosing $h$ small enough we can construct a composite where pulses travel arbitrarily much faster than they do in either of the two phases, but still slower than the speed of light in a vacuum. The volume fraction of the geometry realizing the bound is $\propto \sqrt{h}$. Hence, large relative speed-ups are obtained by doping phase two with phase one with the ratio of the dielectric constants, $h$, being small.

In Figure 1 we plot $\mathcal{G}(h,\beta)$ for a range of parameter values. The dashed line is the bound if we constrain the composite to be isotropic. The bounds in the isotropic cases are realized by geometries for the composites that attain the Hashin-Shtrikman bounds [6]. Note that the bound on the speed-up almost coincides in the isotropic (dashed lines) and anisotropic cases (solid lines) especially for small $h$.

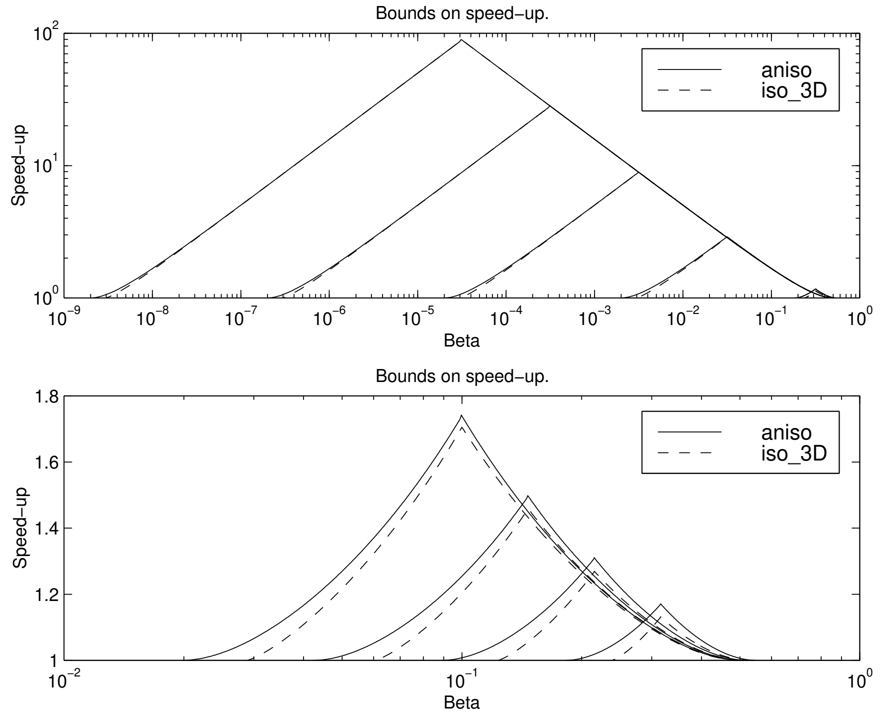


Figure 1: This figure shows the optimal speed-up as a function of $\beta$ for a range of values of the ratio of the dielectric constants $h$. The solid lines correspond to the anisotropic case and the dashed line to the isotropic bound. Note that these almost coincide. In the top plot we use $h \in \{10^{-9},\ 10^{-7},\ 10^{-5},\ 10^{-3},\ 10^{-1}\}$ and obtain the largest speed-ups for small $h$. In the bottom plot we use $h \in \{.01,\ .02,\ .05,\ .1\}$.

## 2.2 Maximal decrease in group velocity

Let as above $\{v_1, v_2, v^*\}$ be respectively the group velocities of the phases and the composite and define

$$J \quad = \quad \min[v_1/v_*, v_2/v_*].$$

The lower bound for the group velocity is obtained by maximizing this quantity with respect to the *geometry* of the composite and the *volume fraction*. We find that the bound for the 'slow-down' is

$$J \quad \leq \quad \mathcal{J}(h,\beta) \quad = \quad \begin{cases} \frac{2(1-h\beta)^{3/2}}{3\sqrt{3(1-\beta)h}(1-h)} & \text{for } \max[0, \frac{3h-1}{2h}] \leq \beta \leq \sqrt{h} \\ \frac{2(1-h\beta)^{3/2}}{3\sqrt{3(1-\beta)}(1-h)\beta} & \text{for } \sqrt{h} \leq \beta \leq \frac{2}{3-h} \\ 1 & \text{else} \end{cases} \tag{9}$$

with $h$ and $\beta$ defined in (6). The bound is again realized by propagation in a laminated medium, but now with lamination *parallel* to the direction of propagation and with the radiation polarized so the electric field is perpendicular to the layer interfaces. The bound when maximized also with respect to the value of $\beta$ becomes

$$\max_{\beta} \mathcal{J}(h,\beta) = \mathcal{J}(h,\sqrt{h}) = \frac{2(1+\sqrt{h}+h)^{3/2}}{\sqrt{h}(1+\sqrt{h})3^{3/2}} \sim \frac{2}{\sqrt{h}3^{3/2}} \qquad as \quad h \downarrow 0. \tag{10}$$

At the maximal slow-down $p = 1 - 2h$ and then $v_1 = v_2$. Note that a large slow-down can be obtained for $h$ small, then the volume fraction for the 'optimal' composite is close to unity. Thus contrary to the maximal speed-up case, a large slow down is obtained when material one is doped with material two, rather than visa versa.

In Figure 2 we plot $\mathcal{J}(h,\beta)$ for a range of parameter values. Large slow downs are obtained when $h$ is small and $\beta$ is close to zero. This corresponds to the dielectric constant of material two being much smaller than that of material one and, moreover, that the relative dispersion in material two is much larger than the relative dispersion in material one.

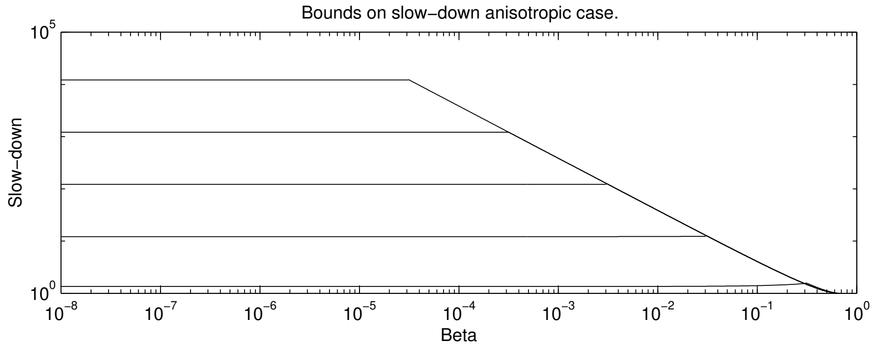


Figure 2: The Figure shows the optimal slow-down as a function of $\beta$ for a range of values of $h$. The set of solid lines correspond to $h \in \{10^{-9},\ 10^{-7},\ 10^{-5},\ 10^{-3},\ 10^{-1}\}$ with the largest slow-downs being obtained for small $h$ values.

# 3 Illustration using Lorentzian phases

Consider the following Lorentzian [5] models for the dielectric constants

$$\varepsilon_1(\omega) = 1 + \frac{\Delta^{-1}}{1-\omega^2} \tag{11}$$

$$\varepsilon_2(\omega) \;=\; 1 + \frac{\sqrt{\Delta}}{1/4 + \sqrt{\Delta} - \omega^2}. \tag{12}$$

with $\Delta = 10^{-3}$. Note that for $\sqrt{\Delta} < \omega < \sqrt{1/4 + 2\sqrt{\Delta}}$ the dielectric constant of phase two, $\varepsilon_2$, is negative and the group velocity, as defined by (2), is not a useful concept (the wave amplitude decays exponentially). We denote the center frequency of the propagating pulse as $\omega_0$ and choose the volume fraction of material one to be $p = \sqrt{h} = \sqrt{\varepsilon_2(\omega_0)/\varepsilon_1(\omega_0)}$. The composite is laminated in the direction orthogonal to propagation such that the effective dielectric constant of the composite is $\varepsilon^* = p\ \varepsilon_1 + (1-p)\ \varepsilon_2$. In Figure 3 we show in the top plot the group velocities as function of $\omega_0$. The group velocity is normalized by the speed of light $c$. The displayed frequency range is close to the resonance frequency of material two. The 'non-propagating' regime corresponds to the frequencies where the group velocity of material two, shown with dashed line, is zero. We let $\Delta = 10^{-3}$, and thus, are doping material two with a very slow material. The bottom plot shows the actual speed-up as function of $\omega_0$. The dashed line is the bound for the speed-up in the small $h$ limit, that is $1/(2h^{1/4})$. Recall that $h$ is the ratio of the dielectric constant of the phases. For the models we use: $h \approx \Delta$. The figure shows that when the group velocities of the phases coincide then the realized speed-up approximately equals the bound. If the group velocity depends sensitively on $\omega_0$ the material is dispersive. From the figure we see that a large speed-up is obtained by combining one phase with large dielectric constant and low dispersion with another phase that has a relatively low dielectric constant and relatively high dispersion. The composite can then be made to exhibit comparatively low dielectric constant and low dispersion and hence a large group velocity.

Next, we illustrate that we can obtain large slow-downs of the composite using the *same* phases. Now we choose $p = 1 - 2h = 1 - 2(\varepsilon_2(\omega_0)/\varepsilon_1(\omega_0))$, thus are doping material one with material two, rather than visa versa. Moreover, we choose the composite to be laminated in the direction orthogonal to the electric field so that the effective dielectric constant of the composite is $1/\varepsilon^* = p/\varepsilon_1 + (1-p)/\varepsilon_2$. The top plot in Figure 4 shows the group velocity for the pure phases and the composite. We show the group-velocity as a function of the center frequency for the pulse, $\omega_0$. The displayed frequency range is close to the resonance frequency of material two. The bottom plot shows the actual slow-down for the composite relative to both of the phases. Note that by combining the same phases as above we are able to construct a composite with comparatively small group velocity. Moreover that the realized slow-down is approximately equal to the bound when the group velocities of the phases coincide. For $\omega_0 \approx 0.5605$ the realized slow down is approximately

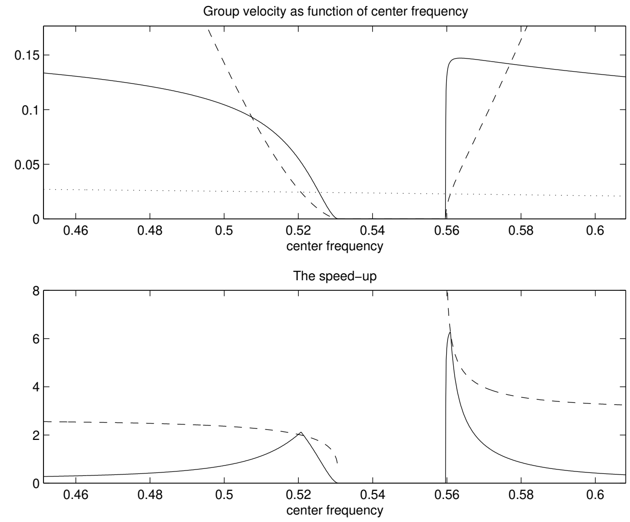


Figure 3: The top plot shows the group velocities as function of center frequency for a composite of two Lorentzian materials. The dotted, dashed and solid lines correspond respectively to material one, two and the composite. In the bottom plot we show the speed-up of the composite relative to the phases. The dashed line is the upper-bound on the speed-up for the given ratio of the dielectric constants. Note that the bound is attained when the group velocities of the phases coincide.

80 !.

# 4 Acknowledgement

The work of GM was supported by the National Science Foundation through grants DMS-9402763 and DMS-9402763 and that of KS was supported by The Research Council of Norway.

# References

[1] Avellaneda M.,A. V. Cherkaev, K. A. Lurie and G. W. Milton, J. Appl. Phys., V 63, (1988), 4989.

[2] Axell J., J. Appl. Phys., V 72, (1992), 1217.

[3] Bergman D. J., Phys. Rev. B, V. 14, (1976), 4304.

[4] Bergman D. J., Phys. Rev. Lett., V 44, (1980), 1285.

[5] Jackson J. D., *Classical Electrodynamics*, John Wiley & Sons, Inc., (1975).

[6] Hashin Z. and S. Shtrikman, J. Appl. Phys., V 33, (1962), 3125.

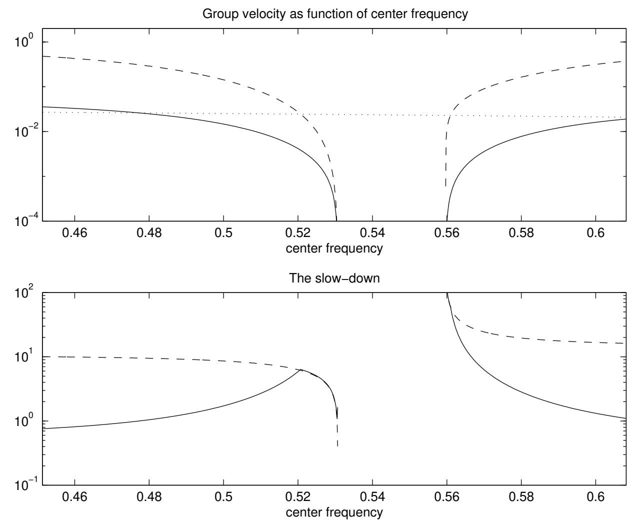


Figure 4: The top plot shows the group velocities as function of center frequency for a composite of materials defined by the Lorentzian model. The dotted, dashed and solid lines correspond respectively to material one, two and the composite. In the bottom plot we show the slow-down of the composite relative to the phases. The dashed line is the upper-bound on the slow-down for the given ratio of the dielectric constants.


[7] Milton G., Appl. Phys. Lett, V 37, (1980), 300.

[8] Milton G. W., J. Apl. Phys., V 52, (1981), 5286.

[9] Milton G. W., J. Appl. Phys, V 52, (1981), 5294.

[10] Milton G. and K. Sølna, *Can mixing materials make electromagnetic signals travel faster?*, preprint, (1999).

[11] Prager S., J. Chem. Phys., V 50, (1969), 4305.