Singular Perturbations in Option Pricing

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Abstract

After the celebrated Black-Scholes formula for pricing options under constant volatility, the need for more complicated nonconstant volatility models in financial mathematics has been the motivation of numerous works during the nineties. In particular, a lot of attention has been paid to stochastic volatility models where the volatility is randomly fluctuating driven by an additional Brownian motion. We have shown in [1, 2] that, in presence of separation of time scales, between the main observed process and the volatility driving process, asymptotic methods are very efficient in capturing the effects of random volatility in simple universal corrections to constant volatility formula. From the point of view of partial differential equations this method corresponds to a singular perturbation analysis. The aim of this paper is to deal with the nonsmoothness of the payoff function inherent to option pricing such as in the case of typical call options. We establish the accuracy of the corrected Black-Scholes price by using an approriate payoff regularization which is removed simultaneously as the asymptotics is performed.

1 Introduction

Stochastic volatility models in financial mathematics can be thought as a Brownian-type particle (the stock price) moving in an environment where the diffusion coefficient is randomly fluctuating in time according to some ergodic (mean-reverting) diffusion process. In the context of Physics there is no natural correlation between these two diffusion processes since they do not "live" in the same space: the second appears as a random coefficient in a diffusion equation which describes the evolution of the probability distribution of the first one. In the context of Finance these two processes, stock price and volatility, jointly define the dynamics of the stock price under

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its physical probability measure or an equivalent risk-neutral martingale measure. Correlation between them is perfectly natural and in fact much needed to account for the so-called leverage effect. The diffusion equation appears as a contingent claim pricing equation, its terminal condition being the payoff of the claim. We refer to [4] or [5] for surveys on stochastic volatility. When volatility is fast mean-reverting, on a time-scale smaller than typical maturities, one can perform an asymptotic singular perturbation analysis of the pricing PDE. As we have shown in [1], this expansion reveals a first correction proportional to the correlation coefficient between the two Brownian motions involved. We refer to [1] for a detailed account of evidence of fast scale in volatility and the use of this asymptotics to parametrize the evolution of the *skew* or the implied volatility surface. We also refer to [3] for a different type of application, namely variance reduction in Monte Carlo methods.

The present paper deals with the accuracy of such an expansion in presence of another essential characteristic feature in option pricing, namely the nonsmoothness of payoff functions such as in the case of put or call options. In [1] we have shown that the leading order term in this expansion corresponds to a Black-Scholes price computed under a constant effective volatility. The first correction involves derivatives of this constant volatility price. When the payoff is smooth we have shown that the corrected price, leading order term plus first correction, has the expected accuracy, namely the remainder of the expansion is of the next order. The nonsmoothness of the payoff, such as for a call option, creates a singularity at the maturity time near the stike price of the option. This paper is devoted to the proof of the accuracy of the approximation in that case. It is important because this is a natural situation in financial mathematics one has to deal with. The proof given here relies on a payoff smoothing argument which can certainly be useful in other contexts.

In Section 2 we introduce the class of stochastic volatility models which we consider. They are written directly under the pricing equivalent martingale measure and with a small parameter representing the short time-scale of volatility. We recall how option prices are given as expected values of discounted payoffs or as solutions of pricing backward parabolic PDE's with terminal conditions at maturity times. In Section 3 we recall the formal asymptotic expansion presented in [1]. In Section 4 we introduce the regularization of the payoff and decompose the main result, accuracy of the price approximation, into three lemmas. Section 5 is devoted to the proof of these lemmas. Detailed computations involving derivatives of Black-Scholes prices up to order seven are given in Appendix where we also recall the properties of the solutions of Poisson equations associated to the infinitesimal generator of the Ornstein-Uhlenbeck process driving the volatility.

2 Class of Models and Pricing Equations

The family of Ornstein-Uhlenbeck driven stochastic volatility models $(S_t^\varepsilon, Y_t^\varepsilon)$ that we consider can be written, under a risk-neutral probability $I\!P^\star$, in terms of the small parameter ε

$$
\begin{aligned}
dS_t^\varepsilon &= rS_t^\varepsilon dt + f(Y_t^\varepsilon)S_t^\varepsilon dW_t^\star, \\
dY_t^\varepsilon &= \left[\frac{1}{\varepsilon}(m - Y_t^\varepsilon) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(Y_t^\varepsilon)\right] dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} d\hat{Z}_t^\star,
\end{aligned}
$$

motions $(W_t^\star, \hat{Z}_t^\star)$ have instantaneous correlation $\rho \in (-1, 1)$:

$$
d\langle W^\star, \hat{Z}^\star\rangle_t = I\!E^\star\{dW_t^\star\, d\hat{Z}_t^\star)\} = \rho\, dt,
$$

and

$$
\Lambda(y) = \frac{\rho(\mu - r)}{f(y)} + \gamma(y)\sqrt{1 - \rho^2},
$$

is a combined market price of risk. The price of the underlying stock is S_t^ε and the volatility is a function f of the process Y_t^ε. At the leading order $1/\varepsilon$, that is omitting the Λ-term, Y_t^ε is an Ornstein-Uhlenbeck (OU) process which is fast mean-reverting with a normal invariant distribution $\mathcal{N}(m, \nu^2)$.

In this *fast mean-reverting* stochastic volatility scenario, the volatility level fluctuates randomly around its mean level, and the epochs of high/low volatility are relatively short. This is the regime that we consider and under which we analyze the price of European derivatives. A derivative is defined by its nonnegative payoff function $H(S)$ and its maturity time T. The payoff function must in general satisfy the integrability condition

$$
I\!E^\star\{H(S_T)^2\} < \infty,
$$

with $I\!E^\star$ denoting expectation with respect to $I\!P^\star$. Moreover, we *assume*:

1. T he volatility is positive and bounded: there are constants m_1 and m_2 such that
$$
0 < m_1 \leq f(y) \leq m_2 < \infty \qquad \forall y \in \mathcal{R}.
$$

2. The volatility risk-premium is bounded:
$$
|\gamma(y)| < l < \infty \qquad \forall y \in \mathcal{R}.
$$
for some constant l.

It is convenient at this stage to make the change of variable

$$
X_t^\varepsilon = \log S_t^\varepsilon, \qquad t \geq 0,
$$

and write the problem in terms of the processes $(X_t^\varepsilon, Y_t^\varepsilon)$ which satisfy, by Itô's formula the stochastic differential equations

$$
\begin{aligned}
dX_t^\varepsilon &= \left(r - \frac{1}{2}f(Y_t^\varepsilon)^2\right) dt + f(Y_t^\varepsilon)\, dW_t^\star, &\quad (2.1)\\
dY_t^\varepsilon &= \left[\frac{1}{\varepsilon}(m - Y_t^\varepsilon) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(Y_t^\varepsilon)\right] dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\, d\hat{Z}_t^\star. &\quad (2.2)
\end{aligned}
$$

We also define the payoff function h in terms of the log stock price via

$$H(e^x) = h(x), \qquad x \in \mathcal{R}.$$

The price at time $t < T$ of this derivative is a function of the present value of the stock price, or equivalently the log stock price, $X_t^\varepsilon = x$ and the present value $Y_t^\varepsilon = y$ of the process driving the volatility. We denote this price by $P^\varepsilon(t,x,y)$. Under the risk-neutral probability measure $I\!P^\star$, it is given by:

$$P^\varepsilon(t,x,y) = I\!E^\star \left\{ e^{-r(T-t)} h(X_T^\varepsilon) | X_t^\varepsilon = x, Y_t^\varepsilon = y \right\}.$$

We shall also write these conditional expectations more compactly as

$$P^\varepsilon(t,x,y) = I\!E^\star{}_{t,x,y} \left\{ e^{-r(T-t)} h(X_T^\varepsilon) \right\}.$$

Under the assumptions on the models considered and the payoff, $P^\varepsilon(t,x,y)$ is the unique classical solution to the associated backward Feynman-Kac partial differential equation problem

$$\begin{aligned} \left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 \right) P^\varepsilon &= 0, \\ P^\varepsilon(T,x,y) &= h(x) \end{aligned} \tag{2.3}$$

in $t < T$, $x, y \in \mathcal{R}$, where we have defined the operators

$$\mathcal{L}_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}, \tag{2.4}$$

$$\mathcal{L}_1 = \sqrt{2}\rho\nu f(y)\frac{\partial^2}{\partial x \partial y} - \sqrt{2}\nu\Lambda(y)\frac{\partial}{\partial y}, \tag{2.5}$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2}f(y)^2\frac{\partial^2}{\partial x^2} + \left(r - \frac{1}{2}f(y)^2 \right)\frac{\partial}{\partial x} - r\,\cdot\,. \tag{2.6}$$

The operator $\mathcal{L}_0$ is the infinitesimal generator of the OU process

$$dY_t = (m - Y_t)\,dt + \nu\sqrt{2}\,d\hat{Z}_t^\star, \tag{2.7}$$

$\mathcal{L}_2$ is the Black-Scholes operator in the log variable and with volatility $f(y)$, and $\mathcal{L}_1$ contains the mixed partial derivative due to the correlation and the derivative due to the market price of risk.

3 Price approximation

We present here the formal asymptotic expansion computed as in [1, 2] which leads to a (first-order in ε) approximation $P^\varepsilon(t,x,y) \approx Q^\varepsilon(t,x)$. In the next section we prove the convergence and accuracy as $\varepsilon \downarrow 0$ of this approximation which consists of the first two terms of the asymptotic price expansion:

$$Q^\varepsilon(t,x) = P_0(t,x) + \sqrt{\varepsilon}P_1(t,x),$$

which do not depend on y and are derived as follows. We start by writing

$$P^\varepsilon = Q^\varepsilon + \varepsilon Q_2 + \varepsilon^{3/2} Q_3 + \cdots = P_0 + \sqrt{\varepsilon} P_1 + \varepsilon Q_2 + \varepsilon^{3/2} Q_3 + \cdots , \tag{3.1}$$

Substituting (3.1) into (2.3) leads to

$$\begin{aligned} &\frac{1}{\varepsilon}\mathcal{L}_0 P_0 + \frac{1}{\sqrt{\varepsilon}}\left(\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0\right) \\ &\quad + \left(\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0\right) + \sqrt{\varepsilon}\left(\mathcal{L}_0 Q_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1\right) + \cdots = 0. \end{aligned} \tag{3.2}$$

We shall next obtain expressions for P_0 and P_1 by successively equating the four leading order terms in (3.2) to zero. We let $\langle\cdot\rangle$ denote the averaging with respect to the invariant distribution $\mathcal{N}(m, \nu^2)$ of the OU process Y:

$$\langle g\rangle = \frac{1}{\nu\sqrt{2\pi}}\int_{\mathcal{R}} g(y) e^{-(m-y)^2/2\nu^2}\, dy. \tag{3.3}$$

Below, we will need to solve the *Poisson equation* associated with $\mathcal{L}_0$:

$$\mathcal{L}_0 \chi + g = 0, \tag{3.4}$$

which requires the solvability condition

$$\langle g\rangle = 0, \tag{3.5}$$

in order to admit solutions with reasonable growth at infinity. Properties of this equation and its solutions are recalled in the Appendix C.

Consider first the leading order term:

$$\mathcal{L}_0 P_0 = 0.$$

Since $\mathcal{L}_0$ takes y-derivatives, we seek solutions which are independent of y: $P_0 = P_0(t, x)$ with the terminal condition $P(T, x) = h(x)$.

Consider next:

$$\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0.$$

Since $\mathcal{L}_1$ contains only terms with derivatives in y it reduces to $\mathcal{L}_0 P_1 = 0$ and we seek again a function $P_1 = P_1(t, x)$, independent of y, with a zero terminal condition $P_1(T, x) = 0$. Hence, $Q^\varepsilon = P_0 + \sqrt{\varepsilon} P_1$, the leading order approximation, does not depend on the current value of the volatility level.

The next equation

$$\mathcal{L}_0 Q_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0 = 0$$

reduces to the Poisson equation $\mathcal{L}_0 Q_2 + \mathcal{L}_2 P_0 = 0$, and its solvability condition

$$\langle \mathcal{L}_2 P_0\rangle = \langle \mathcal{L}_2\rangle P_0 = 0, \tag{3.6}$$

is the Black-Scholes PDE with constant square volatility $\langle f^2 \rangle$ and payoff h. We choose $P_0(t,x)$ to be the classical Black-Scholes price, solution of (3.6) with the terminal condition $P(T,x) = h(x)$.

Observe that $Q_2 = -\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle)P_0$ and consider finally:

$$\mathcal{L}_0 Q_3 + \mathcal{L}_1 Q_2 + \mathcal{L}_2 P_1 = 0. \tag{3.7}$$

It is a Poisson equation in P_3, and its solvability condition gives

$$\langle \mathcal{L}_2 \rangle P_1 = -\langle \mathcal{L}_1 Q_2 \rangle = \langle \mathcal{L}_1 \mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle P_0,$$

which, with its zero terminal condition, determines P_1 as a solution of a Black-Scholes equation with constant square volatility $\langle f^2 \rangle$ and a source. Using the expressions for $\mathcal{L}_i$ one can rewrite the source as:

$$\begin{aligned}
\langle \mathcal{L}_1 \mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle &= \left\langle \mathcal{L}_1 \mathcal{L}_0^{-1} \left(f(y)^2 - \langle f^2 \rangle \right) \right\rangle \frac{1}{2} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) P_0 \\
&= \left(v_3 \frac{\partial^3}{\partial x^3} + (v_2 - 3v_3) \frac{\partial^2}{\partial x^2} + (2v_3 - v_2) \frac{\partial}{\partial x} \right) P_0,
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
v_2 &= \frac{\nu}{\sqrt{2}} (2\rho \langle f \phi' \rangle - \langle \Lambda \phi' \rangle) \\
v_3 &= \frac{\rho \nu}{\sqrt{2}} \langle f \phi' \rangle,
\end{aligned} \tag{3.9}$$

and ϕ is a solution of the Poisson equation:

$$\mathcal{L}_0 \phi(y) = f(y)^2 - \langle f^2 \rangle. \tag{3.10}$$

We can therefore conclude:

1. The first term P_0 is chosen to be the solution of the "homogenized" PDE problem

   $$\begin{aligned}
   \langle \mathcal{L}_2 \rangle P_0 &= 0, \\
   P_0(t,x) &= h(x),
   \end{aligned}$$

   with

   $$\langle \mathcal{L}_2 \rangle = \frac{\partial}{\partial t} + \frac{1}{2} \bar{\sigma}^2 \frac{\partial^2}{\partial x^2} + \left(r - \frac{1}{2} \bar{\sigma}^2 \right) \frac{\partial}{\partial x} - r\cdot,$$

   and where

   $$\bar{\sigma}^2 = \langle f^2 \rangle.$$

   In other words, P_0 is simply the Black-Scholes price of the derivative computed with the effective volatility $\bar{\sigma}$.

2. The second term, or correction to the Black-Scholes price, is chosen to be given explicitly, as a linear combination of the first three derivatives of P_0, by

$$\sqrt{\varepsilon}\,P_1 = -(T-t)\left(V_3^\varepsilon \frac{\partial^3}{\partial x^3} + (V_2^\varepsilon - 3V_3^\varepsilon)\frac{\partial^2}{\partial x^2} + (2V_3^\varepsilon - V_2^\varepsilon)\frac{\partial}{\partial x}\right) P_0, \quad (3.11)$$

with

$$V_{2,3}^\varepsilon = \sqrt{\varepsilon}\,v_{2,3}, \quad (3.12)$$

since it is easily seen, by using $\langle \mathcal{L}_2 \rangle P_0 = 0$, that equation (3.8) is satisfied, and that, on the other hand, the terminal condition $P_1(T,x) = 0$ is clearly satisfied.

Essential instruments in financial markets are put and call options for which the payoff function $H(S)$ is piecewise linear. We shall focus on call options:

$$H(S) = (S-K)^+ \qquad \Rightarrow \qquad h(x) = (e^x - K)^+,$$

for some given strike price $K > 0$. Notice that h is only $\mathcal{C}^0$ smooth with a discontinuous first derivative at the kink $x = \log K$, (at the money in financial terms). Nonetheless, at $t < T$, the Black-Scholes pricing function $P_0(t,x)$ is smooth and $P_1(t,x)$ is well-defined, but second and higher derivatives of P_0 with respect to x blow up as $t \to T$ (at the money).

Our main result on the accuracy of the approximation $Q^\varepsilon = P_0 + \sqrt{\varepsilon}\,P_1$ is as follows:

Theorem 1 *Under the assumptions (1) and (2) above, at a fixed point $t < T$, $x, y \in \mathcal{R}$,*

$$\lim_{\varepsilon \downarrow 0} \frac{|P^\varepsilon(t,x,y) - Q^\varepsilon(t,x)|}{\varepsilon^{1-p}} = 0,$$

for any $p > 0$.

4 Accuracy of the price approximation

In order to prove Theorem 1 we introduce in the next section the regularized price, $P^{\varepsilon,\delta}$, the price with a slightly smoothed payoff with δ being the (small) smoothing parameter. We denote the associated price approximation $Q^{\varepsilon,\delta}$. The proof then involves showing that (i) $P^\varepsilon \approx P^{\varepsilon,\delta}$, (ii) $P^{\varepsilon,\delta} \approx Q^{\varepsilon,\delta}$, (iii) $Q^{\varepsilon,\delta} \approx Q^\varepsilon$, and controlling the accuracy in these approximations by choosing δ appropriately.

4.1 Regularization

We begin by regularizing the payoff, which is a **call option**, by replacing it with the Black-Scholes price of a call with volatility $\bar{\sigma}$ and time to maturity δ. We define

$$h^\delta(x) := C_{BS}(T-\delta, x; K, T; \bar{\sigma}),$$

where $C_{BS}(t, x; K, T; \bar{\sigma})$ denotes the Black-Scholes call option price as a function of current time t, log stock price x, strike price K, expiration date T and volatility $\bar{\sigma}$. It is given by

$$
\begin{aligned}
C_{BS}(t, x; K, T; \bar{\sigma}) &= P_0(t, x; K, T; \bar{\sigma}) = e^x N(d_1) - K e^{-r\frac{\tau^2}{\bar{\sigma}^2}} N(d_2) \qquad (4.1) \\
N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy \\
d_1 &= \frac{x - \log K}{\tau} + b\tau \\
d_2 &= d_1 - \tau,
\end{aligned}
$$

where we define

$$
\tau = \bar{\sigma}\sqrt{T - t} \qquad b = \frac{r}{\bar{\sigma}^2} + \frac{1}{2}.
$$

For $\delta > 0$, this new payoff is $\mathcal{C}^\infty$. The price $P^{\varepsilon,\delta}(t, x, y)$ of the option with the regularized payoff solves

$$
\begin{aligned}
\mathcal{L}^\varepsilon P^{\varepsilon,\delta} &= 0 \\
P^{\varepsilon,\delta}(T, x, y) &= h^\delta(x).
\end{aligned}
$$

4.2 Main convergence result

Let $Q^{\varepsilon,\delta}(t, x)$ denote the first-order approximation to the regularized option price:

$$
P^{\varepsilon,\delta} \approx Q^{\varepsilon,\delta} \equiv P_0^\delta + \sqrt{\varepsilon} P_1^\delta,
$$

where

$$
\begin{aligned}
P_0^\delta(t, x) &= C_{BS}(t - \delta, x; K, T; \bar{\sigma}) \qquad (4.2) \\
\sqrt{\varepsilon} P_1^\delta &= -(T - t)\left(V_3^\varepsilon \frac{\partial^3}{\partial x^3} + (V_2^\varepsilon - 3V_3^\varepsilon)\frac{\partial^2}{\partial x^2} + (2V_3^\varepsilon - V_2^\varepsilon)\frac{\partial}{\partial x}\right) P_0^\delta. \qquad (4.3)
\end{aligned}
$$

We establish the following pathway to proving Theorem 1 where constants may depend on (t, T, x, y) but not on (ε, δ):

Lemma 1 *Fix the point (t, x, y) where $t < T$. There exists constants $\bar{\delta}_1 > 0$, $\bar{\varepsilon}_1 > 0$ and $c_1 > 0$ such that*

$$
|P^\varepsilon(t, x, y) - P^{\varepsilon,\delta}(t, x, y)| \le c_1 \delta
$$

for all $0 < \delta < \bar{\delta}_1$ and $0 < \varepsilon < \bar{\varepsilon}_1$.

This establishes that the solutions to the regularized and unregularized problems are close.

Lemma 2 *Fix the point (t, x, y) where $t < T$. There exists constants $\bar{\delta}_2 > 0$, $\bar{\varepsilon}_2 > 0$ and $c_2 > 0$ such that*

$$
|Q^\varepsilon(t, x) - Q^{\varepsilon,\delta}(t, x)| \le c_2 \delta
$$

for all $0 < \delta < \bar{\delta}_2$ and $0 < \varepsilon < \bar{\varepsilon}_2$.

This establishes that the first-order asymptotic approximations to the regularized and unregularized problems are close.

Lemma 3 *Fix the point (t, x, y) where $t < T$. There exist constants $\bar{\delta}_3 > 0$, $\bar{\varepsilon}_3 > 0$ and $c_3 > 0$ such that*

$$|P^{\varepsilon,\delta}(t,x,y) - Q^{\varepsilon,\delta}(t,x)| \leq c_3 \left(\varepsilon|\log\delta| + \varepsilon\sqrt{\frac{\varepsilon}{\delta}} + \varepsilon\right),$$

for all $0 < \delta < \bar{\delta}_3$ and $0 < \varepsilon < \bar{\varepsilon}_3$.

This establishes that for fixed δ, the approximation to the regularized problem converges to the regularized price as $\varepsilon \downarrow 0$.

The convergence result follows from these lemmas:

Proof of Theorem 1. Take $\bar{\delta} = \min(\bar{\delta}_1, \bar{\delta}_2, \bar{\delta}_3)$ and $\bar{\varepsilon} = \min(\bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\varepsilon}_3)$. Then using lemmas 1, 2 and 3, we obtain

$$\begin{aligned} |P^\varepsilon - Q^\varepsilon| &\leq |P^\varepsilon - P^{\varepsilon,\delta}| + |P^{\varepsilon,\delta} - Q^{\varepsilon,\delta}| + |Q^\varepsilon - Q^{\varepsilon,\delta}| \\ &\leq 2\max(c_1, c_2)\delta + c_3\left(\varepsilon|\log\delta| + \varepsilon\sqrt{\frac{\varepsilon}{\delta}} + \varepsilon\right), \end{aligned}$$

for $0 < \delta < \bar{\delta}$ and $0 < \varepsilon < \bar{\varepsilon}$, where the functions are evaluated at the fixed (t, x, y). Taking $\delta = \varepsilon$, we have

$$|P^\varepsilon - Q^\varepsilon| \leq c_5(\varepsilon + \varepsilon|\log\varepsilon|),$$

for some fixed $c_5 > 0$. It follows that

$$\lim_{\varepsilon\downarrow 0} \frac{|P^\varepsilon(t,x,y) - Q^\varepsilon(t,x)|}{\varepsilon^{1-p}} = 0,$$

for any $p > 0$.

5 Proof of lemmas

5.1 Proof of Lemma 1

We use the probabilistic representation of the price given as the expected discounted payoff with respect to the risk-neutral pricing equivalent martingale measure $I\!P^\star$.

$$P^{\varepsilon,\delta}(t,x,y) = I\!E^\star{}_{t,x,y}\left\{e^{-r(T-t)}h^\delta(X_T^\varepsilon)\right\},$$

and define the new process $(\tilde{X}_t^\varepsilon)$ by

$$d\tilde{X}_t^\varepsilon = \left(r - \frac{1}{2}\tilde{f}(t, Y_t^\varepsilon)^2\right)dt + \tilde{f}(t, Y_t^\varepsilon)\left(\sqrt{1-\rho^2}\, d\hat{W}_t^\star + \rho\, d\hat{Z}_t^\star\right),$$

where $(\hat{W}_t^\star)$ is a Brownian motion independent of $(\hat{Z}_t^\star)$, (Y_t^ε) is still a solution of (2.2) and

$$\tilde{f}(t,y) = \begin{cases} f(y) & \text{for } t \leq T \\ \bar{\sigma} & \text{for } t > T. \end{cases}$$

Then we can write

$$P^{\varepsilon,\delta}(t,x,y) = I\!E^{\star}{}_{t,x,y}\left\{e^{-r(T-t+\delta)}h(\tilde{X}^{\varepsilon}_{T+\delta})\right\},$$

and

$$P^{\varepsilon}(t,x,y) = I\!E^{\star}{}_{t,x,y}\left\{e^{-r(T-t)}h(\tilde{X}^{\varepsilon}_{T})\right\}.$$

Next we use the iterated expectations formula

$$P^{\varepsilon,\delta}(t,x,y) - P^{\varepsilon}(t,x,y) = I\!E^{\star}{}_{t,x,y}\left\{I\!E^{\star}\left\{e^{-(T-t+\delta)}h(\tilde{X}^{\varepsilon}_{T+\delta}) - e^{-(T-t)}h(\tilde{X}^{\varepsilon}_{T}) \mid (\hat{Z}^{\star}_s)_{t\leq s\leq T}\right\}\right\}$$

to obtain a representation of this price difference in terms of the Black-Scholes function P_0 which is smooth away from the terminal date T. In the uncorrelated case it corresponds to the Hull-White formula [6]. In the correlated case, as considered here, this formula is in [7], and can be found in [1](2.8.3). Conditioned on the path of the second Brownian motion, it is simple to compute explicitly

$$\tilde{X}^{\varepsilon}_{T} \mid (\hat{Z}^{\star}_s)_{t\leq s\leq T} \sim \mathcal{N}(m^{\varepsilon}_1, v^{\varepsilon}_1),$$

where the mean and variance are given by

$$\begin{aligned} m^{\varepsilon}_1 &= x + \xi_{t,T} + (r - \frac{1}{2}\bar{\sigma}^2_{\rho})(T-t) \\ v^{\varepsilon}_1 &= \bar{\sigma}^2_{\rho}(T-t) \end{aligned}$$

and we define

$$\begin{aligned} \xi_{t,T} &= \rho\int_t^T \tilde{f}(t,Y^{\varepsilon}_s)\,d\hat{Z}^{\star}_s - \frac{1}{2}\rho^2\int_t^T \tilde{f}(t,Y^{\varepsilon}_s)^2 ds \qquad (5.1) \\ \bar{\sigma}^2_{\rho} &= \frac{1-\rho^2}{T-t}\int_t^T \tilde{f}(t,Y^{\varepsilon}_s)^2 ds. \end{aligned}$$

It follows from the calculation that leads to the Black-Scholes formula that

$$I\!E^{\star}\{e^{-(T-t)}h(\tilde{X}^{\varepsilon}_{T}) \mid (\hat{Z}^{\star}_s)_{t\leq s\leq T}\} = P_0(t, \tilde{X}^{\varepsilon}_t + \xi_{t,T}; K, T; \bar{\sigma}_{\rho}).$$

Similarly, we compute

$$\tilde{X}^{\varepsilon}_{T+\delta} \mid (\hat{Z}^{\star}_s)_{t\leq s\leq T} \sim \mathcal{N}(m^{\varepsilon}_2, v^{\varepsilon}_2),$$

where the mean and variance are given by

$$\begin{aligned} m^{\varepsilon}_2 &= x + \xi_{t,T} + r\delta + (r - \frac{1}{2}\tilde{\sigma}^2_{\rho,\delta})(T-t) \\ v^{\varepsilon}_2 &= \tilde{\sigma}^2_{\rho,\delta}(T-t), \end{aligned}$$

and we define

$$\tilde{\sigma}^2_{\rho,\delta} = \bar{\sigma}^2_{\rho} + \frac{\delta\bar{\sigma}^2}{T-t}.$$

Therefore

$$\mathbb{E}^\star\{e^{-(T-t+\delta)}h(\tilde{X}^\varepsilon_{T+\delta}) \mid (\hat{Z}^\star_s)_{t\le s\le T}\} = P_0(t, \tilde{X}^\varepsilon_t + \xi_{t,T} + r\delta; K, T; \tilde{\sigma}_{\rho,\delta}),$$

and we can write

$$P^{\varepsilon,\delta}(t,x,y) - P^\varepsilon(t,x,y) = \mathbb{E}^\star{}_{t,x,y}\left\{P_0(t, x+\xi_{t,T}+r\delta; K, T; \tilde{\sigma}_{\rho,\delta}) - P_0(t, x+\xi_{t,T}; K, T; \bar{\sigma}_\rho)\right\}.$$

Using the explicit representation (4.1) and that $\bar{\sigma}_\rho$ is bounded above and below as $f(y)$ is, we find

$$|P_0(t, x+\xi_{t,T}+r\delta; K, T; \tilde{\sigma}_{\rho,\delta}) - P_0(t, x+\xi_{t,T}; K, T; \bar{\sigma}_\rho)| \le \delta c_1 (e^{\xi_{t,T}}\left[|\xi_{t,T}|+1\right]+1)$$

for some c_1 and for δ small enough. Using the definition (5.1) of $\xi_{t,T}$ and the existence of its exponential moments, we thus find that

$$|P^\varepsilon(t,x,y) - P^{\varepsilon,\delta}(t,x,y)| \le c_2\delta$$

for some c_2 and for δ small enough.

5.2 Proof of Lemma 2

From the definition (3.11) of the correction $\sqrt{\varepsilon}P_1$ and the corresponding definition (4.3) of the correction $\sqrt{\varepsilon}P_1^\delta$ we deduce

$$\begin{aligned} &Q^{\varepsilon,\delta} - Q^\varepsilon \\ &= \left(1-(T-t)\left(V_3^\varepsilon\frac{\partial^3}{\partial x^3} + (V_2^\varepsilon - 3V_3^\varepsilon)\frac{\partial^2}{\partial x^2} + (2V_3^\varepsilon - V_2^\varepsilon)\frac{\partial}{\partial x}\right)\right)(P_0^\delta - P_0). \end{aligned}$$

From the definition (3.9) of the v_i's, the definition (3.12) of the V_i's and the bounds on the solution of the Poisson equation (3.10) given in Appendix C, it follows that

$$\max(|V_2^\varepsilon|, |V_3^\varepsilon|) \le c_1\sqrt{\varepsilon}$$

for some constant $c_1 > 0$. Notice that we can write

$$P_0^\delta(t,x) = P_0(t-\delta, x).$$

Using the explicit formula (4.1), it is easily seen that P_0 and its successive derivatives with respect to x are differentiable in t at any $t < T$. Therefore we conclude that for (t,x,y) fixed with $t < T$:

$$|Q^\varepsilon(t,x) - Q^{\varepsilon,\delta}(t,x)| \le c_2\delta$$

for some $c_2 > 0$ and δ small enough.

5.3 Proof of Lemma 3

We first introduce some additional notation. Define the error $Z^{\varepsilon,\delta}$ in the approximation for the regularized problem by

$$P^{\varepsilon,\delta} = P_0^\delta + \sqrt{\varepsilon} P_1^\delta + \varepsilon Q_2^\delta + \varepsilon^{3/2} Q_3^\delta - Z^{\varepsilon,\delta},$$

for Q_2^δ and Q_3^δ stated below in (5.3) and (5.4). It follows that

$$\begin{aligned}
\mathcal{L}^\varepsilon Z^{\varepsilon,\delta} &= \mathcal{L}^\varepsilon \left(P_0^\delta + \sqrt{\varepsilon} P_1^\delta + \varepsilon Q_2^\delta + \varepsilon^{3/2} Q_3^\delta - P^{\varepsilon,\delta} \right) \\
&= \frac{1}{\varepsilon} \mathcal{L}_0 P_0^\delta + \frac{1}{\sqrt{\varepsilon}} (\mathcal{L}_0 P_1^\delta + \mathcal{L}_1 P_0^\delta) \\
&\quad + (\mathcal{L}_0 Q_2^\delta + \mathcal{L}_1 P_1^\delta + \mathcal{L}_2 P_0^\delta) + \sqrt{\varepsilon} \left(\mathcal{L}_0 Q_3^\delta + \mathcal{L}_1 Q_2^\delta + \mathcal{L}_2 P_1^\delta \right) \\
&\quad + \varepsilon \left(\mathcal{L}_1 Q_3^\delta + \mathcal{L}_2 Q_2^\delta + \sqrt{\varepsilon} \mathcal{L}_2 Q_3^\delta \right) \\
&= \varepsilon \left(\mathcal{L}_1 Q_3^\delta + \mathcal{L}_2 Q_2^\delta \right) + \varepsilon^{3/2} \mathcal{L}_2 Q_3^\delta \equiv G^{\varepsilon,\delta}
\end{aligned} \tag{5.2}$$

because $P^{\varepsilon,\delta}$ solves the original equation $\mathcal{L}^\varepsilon P^{\varepsilon,\delta} = 0$ and we choose P_0^δ, P_1^δ, Q_2^δ and Q_3^δ to cancel the first four terms. In particular, we choose

$$Q_2^\delta(t,x,y) = -\frac{1}{2} \phi(y) \left(\frac{\partial^2 P_0^\delta}{\partial x^2} - \frac{\partial P_0^\delta}{\partial x} \right), \tag{5.3}$$

so that

$$\mathcal{L}_0 Q_2^\delta = -\mathcal{L}_2 P_0^\delta,$$

(with an "integration constant" arbitrarily set to zero) whereas Q_3^δ is a solution of the Poisson equation

$$\mathcal{L}_0 Q_3^\delta = -(\mathcal{L}_1 Q_2^\delta + \mathcal{L}_2 P_1^\delta), \tag{5.4}$$

where the centering condition is insured by our choice of P_1^δ.
At the terminal time T we have

$$Z^{\varepsilon,\delta}(T,x,y) = \varepsilon \left(Q_2^\delta(T,x,y) + \sqrt{\varepsilon} Q_3^\delta(T,x,y) \right) \equiv H^{\varepsilon,\delta}(x,y),$$

where we have used the terminal conditions $P^{\varepsilon,\delta}(T,x,y) = P_0^\delta(T,x,y) = h^\delta(x)$ and $P_1^\delta(T,x,y) = 0$. This assumes smooth derivatives of P_0^δ in the domain $t \leq T$ which is the case because h^δ is smooth. It is shown in Appendix A that the source term $G^{\varepsilon,\delta}(t,x,y)$ on the right-side of equation (5.2) can be written in the form

$$\begin{aligned}
G^{\varepsilon,\delta} &= \varepsilon \left(\sum_{i=1}^{4} g_i^{(1)}(y) \frac{\partial^i}{\partial x^i} P_0^\delta + (T-t) \sum_{i=1}^{6} g_i^{(2)}(y) \frac{\partial^i}{\partial x^i} P_0^\delta \right) \\
&\quad + \varepsilon^{3/2} \left(\sum_{i=1}^{5} g_i^{(3)}(y) \frac{\partial^i}{\partial x^i} P_0^\delta + (T-t) \sum_{i=1}^{7} g_i^{(4)}(y) \frac{\partial^i}{\partial x^i} P_0^\delta \right).
\end{aligned} \tag{5.5}$$

In Appendix A we also show that the terminal condition $H^{\varepsilon,\delta}(x,y)$ in (5.5) can be written

$$H^{\varepsilon,\delta}(x,y) \;=\; \varepsilon\left(\sum_{i=1}^{2} h_i^{(1)}(y)\frac{\partial^i}{\partial x^i}P_0^\delta(T,x)\right)+\varepsilon^{3/2}\left(\sum_{i=1}^{3} h_i^{(2)}(y)\frac{\partial^i}{\partial x^i}P_0^\delta(T,x)\right) \tag{5.6}$$

To bound the contributions from the source term and terminal conditions we need the following two lemmas that are derived in Appendix C and Appendix B respectively:

Lemma 4 *Let $\chi = g_i^{(j)}$ or $\chi = h_i^{(j)}$ with the functions $g_i^{(j)}$ and $h_i^{(j)}$ being defined in (5.5) and (5.6) then there exists a constant $c > 0$ (which may depend on y) such that $I\!E^\star\left\{|\chi(Y_s^\varepsilon)|Y_t^\varepsilon = y\right\} \le c < \infty$ for $t \le s \le T$.*

Lemma 5 *Assume $T - t > \Delta > 0$ and $I\!E^\star\left\{|\chi(Y_s^\varepsilon)|Y_t^\varepsilon = y\right\} \le c_1 < \infty$ for some constant c_1 then there exists constants $c_2 > 0$ and $\bar\delta > 0$ such that for $\delta < \bar\delta$ and $t \le s \le T$*

$$|I\!E^\star{}_{t,x,y}\left\{\sum_{i=1}^{n}\chi(Y_s^\varepsilon)\frac{\partial^i}{\partial x^i}P_0^\delta(s,X_s^\varepsilon)\right\}| \;\le\; c_2[T+\delta-s]^{\min[0,1-n/2]} \tag{5.7}$$

$$\begin{aligned}
&|I\!E^\star{}_{t,x,y}\left\{\int_t^T (T-s)^p\sum_{i=1}^{n} e^{-r(s-t)}\chi(Y_s^\varepsilon)\frac{\partial^i}{\partial x^i}P_0^\delta(s,X_s^\varepsilon)\; ds\right\}| \qquad (5.8)\\
&\le \begin{cases} c_2\;|\log(\delta)| & \textit{for } n = 4+2p\\ c_2\;\delta^{\min[0,p+(4-n)/2]} & \textit{else}\;. \end{cases}
\end{aligned}$$

Proof of Lemma 3

We use the probabilistic representation of equation (5.2), $\mathcal{L}^\varepsilon Z^{\varepsilon,\delta} = G^{\varepsilon,\delta}$ with terminal condition $H^{\varepsilon,\delta}$:

$$Z^{\varepsilon,\delta}(t,x,y) = I\!E^\star{}_{t,x,y}\left\{e^{-r(T-t)}H^{\varepsilon,\delta}(X_T^\varepsilon,Y_T^\varepsilon) - \int_t^T e^{-r(s-t)}G^{\varepsilon,\delta}(s,X_s^\varepsilon,Y_s^\varepsilon)ds\right\}.$$

From Lemma 5 it follows that there exists a constant $c > 0$ such that

$$|I\!E^\star{}_{t,x,y}\left\{\int_t^T e^{-r(s-t)}G^{\varepsilon,\delta}(X_s^\varepsilon,Y_s^\varepsilon)ds\right\}| \;\le\; c\left\{\varepsilon+\varepsilon|\log(\delta)|+\varepsilon\sqrt{\varepsilon/\delta}\right\} \tag{5.9}$$

$$|I\!E^\star{}_{t,x,y}\left\{H^{\varepsilon,\delta}(X_T^\varepsilon,Y_T^\varepsilon)\right\}| \;\le\; c\left\{\varepsilon+\varepsilon\sqrt{\varepsilon/\delta}\right\}, \tag{5.10}$$

and therefore also for (t,x,y) fixed with $t < T$:

$$\begin{aligned}
|P^{\varepsilon,\delta} - Q^{\varepsilon,\delta}| \;&=\; |\varepsilon Q_2^\delta + \varepsilon^{3/2}Q_3^\delta - Z^{\varepsilon,\delta}|\\
&\le\; c\left\{\varepsilon+\varepsilon|\log(\delta)|+\varepsilon\sqrt{\varepsilon/\delta}\right\}.
\end{aligned}\tag{5.11}$$

since Q_2^δ and Q_3^δ evaluated for $t < T$ can also be bounded using (5.3) and (A.5).

A Expressions for Source Term and Terminal Condition

From (5.2), the source term in the equation for the error $Z^{\varepsilon,\delta}$ is

$$G^{\varepsilon,\delta} = \varepsilon\left(\mathcal{L}_1 Q_3^\delta + \mathcal{L}_2 Q_2^\delta\right) + \varepsilon^{3/2}\mathcal{L}_2 Q_3^\delta. \tag{A.1}$$

To obtain an explicit form for this source term, we consider the three terms separately. We first introduce the convenient notation:

$$\begin{aligned} \mathcal{D} &\equiv \frac{\partial}{\partial x} \\ \mathcal{D}_2 &\equiv \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}. \end{aligned}$$

Consider the term $\mathcal{L}_2 Q_2^\delta$ in (A.1). Using that

$$\begin{aligned} \mathcal{L}_2 &= \mathcal{L}_H(f(y)) = \mathcal{L}_H(\bar\sigma) + \frac{1}{2}\left(f(y)^2 - \bar\sigma^2\right)\mathcal{D}_2 \qquad (A.2) \\ \mathcal{L}_H(\bar\sigma)\mathcal{D}_2 P_0^\delta &= 0, \end{aligned}$$

and (5.3), one deduces:

$$\mathcal{L}_2 Q_2^\delta = -\frac{1}{4}\left(f(y)^2 - \bar\sigma^2\right)\phi(y)\mathcal{D}_2\mathcal{D}_2 P_0^\delta.$$

Consider next the term $\mathcal{L}_1 Q_3^\delta$ in (A.1). We have using (3.7)

$$\begin{aligned} Q_3^\delta &= -\mathcal{L}_0^{-1}\left(\mathcal{L}_1 Q_2^\delta + \mathcal{L}_2 P_1^\delta - \langle \mathcal{L}_1 Q_2^\delta + \mathcal{L}_2 P_1^\delta\rangle\right), \qquad (A.3) \\ &= -\mathcal{L}_0^{-1}\left(\mathcal{L}_1 Q_2^\delta - \langle \mathcal{L}_1 Q_2^\delta\rangle + \left(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle\right) P_1^\delta\right), \end{aligned}$$

(again with an "integration constant" arbitrarily set to zero). It follows from (5.3) that:

$$\begin{aligned} \mathcal{L}_1 Q_2^\delta &= \left(\sqrt{2}\nu\rho f(y)\frac{\partial^2}{\partial x\partial y} - \sqrt{2}\nu\Lambda(y)\frac{\partial}{\partial y}\right)\left(-\frac{1}{2}\phi(y)\mathcal{D}_2 P_0^\delta\right) \\ &= -\frac{1}{\sqrt{2}}\nu\rho f(y)\phi'(y)\mathcal{D}\mathcal{D}_2 P_0^\delta + \frac{1}{\sqrt{2}}\nu\Lambda(y)\phi'(y)\mathcal{D}_2 P_0^\delta. \end{aligned}$$

Now let

$$\begin{aligned} \mathcal{L}_0\psi_1 &= f(y)\phi'(y) - \langle f\phi'\rangle, \qquad (A.4) \\ \mathcal{L}_0\psi_2 &= \Lambda(y)\phi'(y) - \langle \Lambda\phi'\rangle, \end{aligned}$$

then we find using (3.10) and (A.2) that Q_3^δ can be written:

$$Q_3^\delta = \left(\frac{\nu\rho}{\sqrt{2}}\psi_1(y)\mathcal{D}\mathcal{D}_2 P_0^\delta - \frac{\nu}{\sqrt{2}}\psi_2(y)\mathcal{D}_2 P_0^\delta\right) - \frac{1}{2}\left(\phi(y)\mathcal{D}_2 P_1^\delta\right). \tag{A.5}$$

Substituting for $\mathcal{L}_1$ and expanding gives

$$\begin{aligned}\mathcal{L}_1 Q_3^\delta &= \nu^2\rho^2 f(y)\psi_1'(y)\mathcal{D}\mathcal{D}\mathcal{D}_2 P_0^\delta - \nu^2\rho f(y)\psi_2'(y)\mathcal{D}\mathcal{D}_2 P_0^\delta \\ &\quad -\nu^2\rho\Lambda(y)\psi_1'(y)\mathcal{D}\mathcal{D}_2 P_0^\delta + \nu^2\Lambda(y)\psi_2'(y)\mathcal{D}_2 P_0^\delta \\ &\quad -\frac{\nu}{\sqrt{2}}\left(\rho f(y)\phi'(y)\mathcal{D}\mathcal{D}_2 P_1^\delta - \Lambda(y)\phi'(y)\mathcal{D}_2 P_1^\delta\right).\end{aligned}$$

Consider finally the term $\mathcal{L}_2 Q_3^\delta$ in (A.1), we find using (A.2) and (A.5)

$$\begin{aligned}\mathcal{L}_2 Q_3^\delta &= \frac{1}{2}(f(y)^2-\bar{\sigma}^2)\left[\frac{\rho\nu}{\sqrt{2}}\psi_1(y)\mathcal{D}_2\mathcal{D}\mathcal{D}_2 P_0^\delta - \frac{\nu}{\sqrt{2}}\psi_2(y)\mathcal{D}_2\mathcal{D}_2 P_0^\delta - \frac{1}{2}\phi(y)\mathcal{D}_2\mathcal{D}_2 P_1^\delta\right] \\ &\quad -\frac{1}{2}\phi(y)\mathcal{D}_2\left(v_3\mathcal{D}_3 P_0^\delta + v_2\mathcal{D}_2 P_0^\delta\right),\end{aligned}$$

with

$$\mathcal{D}_3 = \frac{\partial^3}{\partial x^3} - 3\frac{\partial^2}{\partial x^2} + 2\frac{\partial}{\partial x}$$

and $v_{2,3}$ defined in (3.9).
To summarize, the source term is given by

$$\begin{aligned}G^{\varepsilon,\delta} &= \varepsilon\left\{\nu^2\rho^2 f(y)\psi_1'(y)\mathcal{D}\mathcal{D}\mathcal{D}_2 P_0^\delta - \nu^2\rho f(y)\psi_2'(y)\mathcal{D}\mathcal{D}_2 P_0^\delta\right. \\ &\quad -\nu^2\rho\Lambda(y)\psi_1'(y)\mathcal{D}\mathcal{D}_2 P_0^\delta + \nu^2\Lambda(y)\psi_2'(y)\mathcal{D}_2 P_0^\delta \\ &\quad -\frac{\nu}{\sqrt{2}}\left(\rho f(y)\phi'(y)\mathcal{D}\mathcal{D}_2 P_1^\delta - \Lambda(y)\phi'(y)\mathcal{D}_2 P_1^\delta\right) \\ &\quad \left.-\frac{1}{4}\left(f(y)^2-\bar{\sigma}^2\right)\phi(y)\mathcal{D}_2\mathcal{D}_2 P_0^\delta\right\} \\ &\quad +\varepsilon^{3/2}\left\{\frac{1}{2}(f(y)^2-\bar{\sigma}^2)\left[\frac{\rho\nu}{\sqrt{2}}\psi_1(y)\mathcal{D}_2\mathcal{D}\mathcal{D}_2 P_0^\delta - \frac{\nu}{\sqrt{2}}\psi_2(y)\mathcal{D}_2\mathcal{D}_2 P_0^\delta - \frac{1}{2}\phi(y)\mathcal{D}_2\mathcal{D}_2 P_1^\delta\right]\right. \\ &\quad \left.-\frac{1}{2}\phi(y)\mathcal{D}_2(v_3\mathcal{D}_3 P_0^\delta + v_2\mathcal{D}_2 P_0^\delta)\right\}\end{aligned}$$

By inspection, this can be written in the form (5.5).

From (5.3) and (A.5) we can also see that the terminal condition H^ε in (5.5) can be written in the form (5.6).

B Proof of Lemma 5

To prove Lemma 5 notice first that a calculation based on the analytic expression for the Black-Scholes price in the standard constant volatility case gives

$$\partial_x^n P_0^\delta(s,x) = \begin{cases} e^x N(u/\tau + b\tau) & \text{for } n=1 \\ e^x N(u/\tau+b\tau) + \sum_{i=0}^{n-2}\frac{b_i^{(n)}}{\tau}e^u\partial_u^i e^{-(u/\tau+b\tau)^2/2} & \text{for } n\geq 2 \end{cases} \tag{B.1}$$

for some constants b_i and with

$$\begin{aligned}\tau &\equiv \bar{\sigma}\sqrt{T+\delta-s} \\ u &\equiv x-\log(K) \\ b &\equiv (r/\bar{\sigma}^2+1/2).\end{aligned}$$

Assume first that $T-s \geq (T-t)/2 > 0$, then $0 < m_1\sqrt{(T-t)/2} \leq \tau \leq m_2\sqrt{T+\delta-t}$ and $0 < b \leq (r/m_1^2+1/2)$. Since $\partial_x^i P_0^\delta(s,x)$ is bounded it follows that

$$\begin{aligned} & |\mathbb{E}^{\star}{}_{t,x,y}\left\{\chi(Y_s^\varepsilon)\partial_x^i P_0^\delta(s,X_s^\varepsilon)\right\}| \\ = & \; |\mathbb{E}^{\star}{}_{t,x,y}\left\{\chi(Y_s^\varepsilon)\mathbb{E}^{\star}{}_{t,x,y}\left\{\partial_x^i P_0^\delta(s,X_s^\varepsilon) \;\mid \hat{Z}_v^\star; t\leq v\leq s\right\}\right\}| \\ \leq & \; c\mathbb{E}^{\star}{}_{t,x,y}\left\{|\chi(Y_s^\varepsilon)|\right\} \end{aligned} \tag{B.2}$$

for some constant c which may depend on x.

Consider next the case $T-s < (T-t)/2 > 0$, then

$$\begin{aligned} & |\mathbb{E}^{\star}{}_{t,x,y}\left\{\frac{1}{\tau}e^u\partial_u^i e^{-(u/\tau+b\tau)^2/2} \mid \hat{Z}_v^\star; t\leq v\leq s\right\}| \\ = & \; \frac{1}{\tau}|\int e^u\partial_u^i e^{-(u/\tau+b\tau)^2/2}p(u)du| \\ = & \; \frac{1}{\tau^i}|\int e^{\tau u}\partial_u^i e^{-(u+b\tau)^2/2}p(\tau u)du| \;\leq\; \frac{c}{\tau^i} \end{aligned} \tag{B.3}$$

where p is the conditional distribution of $u \equiv X_s^\varepsilon - \log(K)$, which is the Gaussian distribution with variance at least $(T-t)(1-\rho^2)m_1^2/2$. Lemma 5 follows readily from (B.2) and (B.3).

C On the solution of the Poisson equation

Let χ solve

$$\mathcal{L}_0\chi + g \;=\; 0,$$

with $\mathcal{L}_0$ defined as in (2.4) and with g satisfying the centering condition

$$\langle g\rangle = 0,$$

where the averaging is done with respect to the invariant distribution associated to the infinitesimal generator $\mathcal{L}_0$ (see (3.3) for an explicit formula). Using the explicit form of the differential operator $\mathcal{L}_0$, one can easily deduce that

$$\Phi(y)\chi^{'}(y) \;=\; \frac{-1}{\nu^2}\int_{-\infty}^{y} g(z)\Phi(z)\;dz = \frac{1}{\nu^2}\int_{y}^{\infty} g(z)\Phi(z)\;dz$$

with Φ being the probability density of the invariant distribution $\mathcal{N}(m,\nu^2)$ associated with $\mathcal{L}_0$. From this it follows that if g is bounded

$$\begin{aligned} |\chi^{'}(y)| &\leq c_1 \\ |\chi(y)| &\leq c_2(1+\log(1+|y|)). \end{aligned}$$

We refer to [1](5.2.2) for details. Notice that χ in Lemma 4 satisfies

$$|\chi(y)| \;\leq\; c\max(|\phi(y)|, |\phi^{'}(y)|, |\psi_{1,2}(y)|, |\psi_{1,2}^{'}(y)|)$$

for some constant c and with ϕ and $\psi_{1,2}$ defined in (3.10) and (A.4) respectively. These functions are solutions of Poisson equations with $g = f^2 - \langle f^2 \rangle$ or $g = f\phi' - \langle f\phi' \rangle$ or $g = \lambda\phi' - \langle \Lambda\phi' \rangle$ which are bounded. Therefore $\chi(y)$ is at most logarithmically growing at infinity. The bound in Lemma 4 now follows from classical a priori estimates on the moments of the process Y_t^ε which are uniform in ε as can easily be seen by a simple time change $t = \varepsilon t'$ in (2.2).

References

[1] J.-P. Fouque, G. Papanicolaou, and K.R. Sircar. *Derivatives in Financial Markets with Stochastic Volatility.* Cambridge University Press, 2000.

[2] J.-P. Fouque, G. Papanicolaou and K.R. Sircar. *Mean-reverting stochastic volatility.* International J. Theor. and Appl. Finance, **13**(2): 101-142. (2000).

[3] J.-P. Fouque and T. Tullie. *Variance Reduction for Monte Carlo Simulation in a Stochastic Volatility Environment.* Quantitative Finance, to appear: 2002.

[4] R. Frey. *Derivative asset analysis in models with level-dependent and stochastic volatility.* CWI Quarterly **10**(1), pp 1-34, 1996.

[5] E. Ghysels, A. Harvey and E. Renault. Stochastic volatility, *in* G. Maddala and C. Rao (eds), *Statistical Methods in Finance*, Vol. 14 of *Handbook of Statistics*, North Holland, Amsterdam, chapter 5, pp 119-191, 1996.

[6] J. Hull. and A. White. *The Pricing of Options on Assets with Stochastic Volatilities.* J. Finance **XLII**(2), pp 281-300, 1987.

[7] G. Willard. *Calculating prices and sensitivities for path-independent derivative securities in multifactor models* PhD thesis, Washington University in St. Louis, MO (1996).