MATH 2B: SAMPLE FINAL #1

- This exam consists of 14 questions and 100 total points.
- Read the directions for each problem carefully and answer all parts of each problem.
- Please show all work needed to arrive at your solutions (unless instructed otherwise). Label graphs and define any notation used. Cross out incorrect scratch-work.
- No calculators or other forms of assistance are allowed. Do not check your cell phones during the exam.
- Clearly indicate your final answer to each problem.

1. (6 points) Suppose that $\int_{-1}^{1} f(x)\,dx = 6$, $\int_{1}^{4} f(x)\,dx = -2$ and $\int_{-1}^{1} h(x)\,dx = 9$. Use this information to compute the following.

a. $\int_4^1 6f(x)\,dx = 6\left[-\int_1^4 f(x)\,dx\right] = -6\cdot(-2) = 12$

b. $\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\cdot 6 + 3\cdot 9 = 12 + 27 = 39$

c. $\int_{-1}^{4} f(x)\,dx = \left(\int_{-1}^{1} + \int_{1}^{4}\right) f(x)\,dx = 6 - 2 = 4$

2. (6 points)

a. Evaluate the following derivative

$$\frac{d}{dx}\int_{\sin(x)}^{x^2} t^3\tan(t)\,dt.$$

$$= 2x\cdot(x^2)^3\tan(x^2) - \cos x\cdot(\sin x)^3\tan(\sin x)$$

$$= 2x^7\tan(x^2) - \cos x\sin^3 x\tan(\sin x)$$

b. Let $r(t)$ be the rate at which the world's oil is consumed, where t is measured in years starting at $t = 0$ representing January 1, 2000, and $r(t)$ is measured in barrels per year. What does $\int_0^{13} r(t)\,dt$ represent and what are its units?

\# barrels of oil consumed between Jan 1st 2000 and Jan 1st 2013 (units = "barrels")

3. (6 points) Evaluate $\int x^2 \tan^{-1} x \, dx$

by parts $u = \tan^{-1} x$, $dv = x^2 dx$

$du = \frac{1}{1+x^2} dx$, $v = \frac{1}{3}x^3$

$$= \frac{1}{3}x^3 \tan^{-1} x - \int \frac{x^3}{3(1+x^2)} dx$$

$$= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3}\int \frac{x(1+x^2) - x}{1+x^2} dx$$

$$= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3}\int x - \frac{x}{1+x^2} dx$$

$$= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + C$$

4. (6 points) Evaluate $\int \frac{1}{x \ln(3x)} dx$

Subs, let $u = \ln(3x) = \ln 3 + \ln x$

$\Rightarrow du = \frac{1}{x} dx$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$= \ln|\ln(3x)| + C.$$

5. (6 points) Evaluate $\int \sin^5(x)\cos^2(x)\,dx$

$$= \int \sin x\,(1-\cos^2 x)^2 \cos^2 x\,dx$$

use $u = \cos x$
$du = -\sin x\,dx$
if you need.

$$= \cancel{\int \sin x\,(\cos^2 x - \cos^4 x)\,dx}$$

$$= \cancel{-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C}$$

$$= \int \sin x\,(\cos^2 x - 2\cos^4 x + \cos^6 x)\,dx$$

$$= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$$

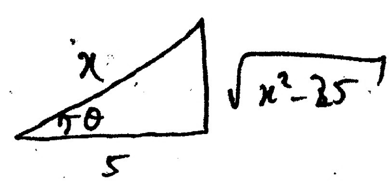
6. (6 points) Evaluate $\int \frac{\sqrt{x^2-25}}{x}\,dx$, where $x > 5$

let $x = 5\sec\theta$
$\Rightarrow dx = 5\sec\theta\tan\theta\,d\theta$

$$\int \frac{5\tan\theta}{5\sec\theta}\cdot 5\sec\theta\tan\theta\,d\theta$$

$$= 5\int \tan^2\theta\,d\theta$$

$$= 5\int \sec^2\theta - 1\,d\theta$$



$$= 5(\tan\theta - \theta) + C$$

$$= \sqrt{x^2-25} - 5\sec^{-1}\frac{x}{5} + C$$

(or $= \sqrt{x^2-25} - 5\cos^{-1}\frac{5}{x} + C$ if you prefer)

7. (6 points) Determine whether each of the following improper integrals are convergent or divergent. Evaluate the integral if it is convergent.

a. $\int_0^2 \frac{1}{(x-2)^2}\,dx$

$$= \lim_{s\to 2^-} \int_0^s (x-2)^{-2}\,dx$$

$$= \lim_{s\to 2^-} \left[-(x-2)^{-1}\right]_0^s$$

$$= \lim_{s\to 2^-} \frac{-1}{s-2} - \frac{1}{2} = +\infty \quad \underline{\text{divergent}}$$

b. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

$$= 2\int_0^\infty \frac{1}{1+x^2}\,dx = 2\lim_{s\to\infty} \tan^{-1}x \Big|_0^s$$

$$= 2\lim_{s\to\infty} \tan^{-1}s = \pi. \quad \underline{\text{convergent}}$$

8. (6 points)

a. Find the average value of the function $f(x) = \sec^2(x)$ on the interval $[0, \frac{\pi}{4}]$.

$$f_{av} = \frac{1}{\frac{\pi}{4}-0}\int_0^{\pi/4} \sec^2 x\,dx = \frac{4}{\pi}\tan x \Big|_0^{\pi/4}$$

$$= \frac{4}{\pi}$$

b. Find the arc length of the curve given by $y = 2x^{3/2}$ from $x = 0$ to $x = 1$.

$$AL = \int_0^1 \sqrt{1+y'^2}\,dx = \int_0^1 \sqrt{1+(3x^{1/2})^2}\,dx$$

$$= \int_0^1 (1+9x)^{1/2}\,dx = \frac{1}{9}\cdot\frac{2}{3}(1+9x)^{3/2}\Big|_0^1$$

$$= \frac{2}{27}\left[10^{3/2}-1\right].$$

9. (6 points) Find the first 5 non-zero terms in the Maclaurin series for $f(x) = (1-x)^{-2}$. Find the associated radius of convergence of this power series.

Method 1

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ has rad of conv 1

$\Rightarrow f(x) = \frac{1}{(1-x)^2} = \frac{d}{dx} \sum_{n=0}^{\infty} x^n$

$= \sum_{n=0}^{\infty} n x^{n-1}$

$= \sum_{m=0}^{\infty} (m+1) x^m$ has rad of conv 1

$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$

Method 2

$f(0) = 1$

$f'(x) = 2(1-x)^{-3} \Rightarrow f'(0) = 2$

$f''(x) = 6(1-x)^{-4} \Rightarrow f''(0) = 6$

$f'''(x) = 24(1-x)^{-5} \Rightarrow f'''(0) = 24$

$f''''(x) = 120(1-x)^{-6} \Rightarrow f''''(0) = 120$

$\Rightarrow$ Series $= 1 + 2x + \frac{6}{2!}x^2 + \frac{24}{3!}x^3 + \frac{120}{4!}x^4 + \dots = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$

Spot pattern: $\sum_{n=0}^{\infty} (n+1) x^n$

Ratio test: $R = \lim_{n\to\infty} \frac{n+1}{n+2} = 1$

10. (6 points) Determine whether each of the following sequences converges or diverges. If it converges, find the limit.

a. $a_n = \left(\frac{2}{3}\right)^n + 3$ Converges to 3. (since $\left(\frac{2}{3}\right)^n \to 0$)

b. $b_n = n^3 e^{-n}$ Converges to 0 (L'Hôpital: $\frac{n^3}{e^n} \to \frac{3n^2}{e^n} \to \frac{6n}{e^n} \to \frac{6}{e^n} \to 0$)

c. $c_n = \tan^{-1}(\ln(n))$ converges to $\frac{\pi}{2}$ ($\ln(n) \xrightarrow[n\to\infty]{} \infty$ and $\tan^{-1} n \xrightarrow[n\to\infty]{} \pi/2$).

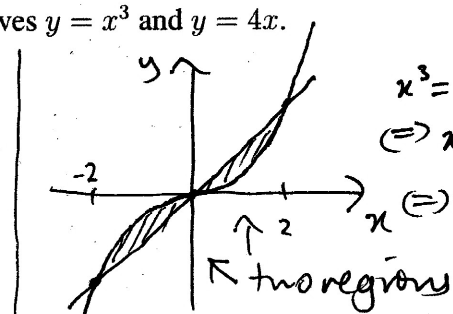
11. (10 points) Find the area of the region(s) bounded by the curves $y = x^3$ and $y = 4x$.

$A = 2\int_0^2 4x - x^3\, dx$

$= 2\left[2x^2 - \frac{1}{4}x^4\right]_0^2$

$= 2\left[2\cdot 4 - \frac{16}{4}\right]$

$= 8$



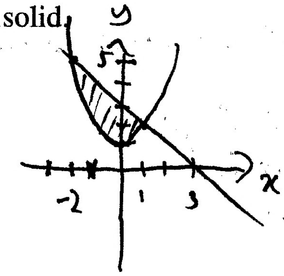
$x^3 = 4x$

$\Leftrightarrow x(x^2-4) = 0$

$\Leftrightarrow x = 0, \pm 2$

12. (10 points)

a. The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the line $y = 5$ to generate a solid. Find the volume of that solid.



$x^2+1=-x+3$

$\Leftrightarrow x^2+x-2=0$

$(x-1)(x+2)$

$\Leftrightarrow x=1,-2$

$r_{out} = 5-(x^2+1) = 4-x^2$

$r_{in} = 5-(-x+3) = 2+x$

$$\Rightarrow V = \pi\int_{-2}^{1} (4-x^2)^2-(2+x)^2\,dx$$

$$= \pi\left[\int_{-2}^{1} 16-8x^2+x^4\,dx - \frac{1}{3}(2+x)^3\Big|_{-2}^{1}\right]$$

$$= \pi\left\{\left[16x-\frac{8}{3}x^3+\frac{1}{5}x^5\right]_{-2}^{1} - \frac{1}{3}\cdot 3^3\right\}$$

$$= \pi\left\{16\cdot 3-\frac{8}{3}(1+8)+\frac{1}{5}(1+32)-9\right\}$$

$$= \pi\left\{48-24+\frac{33}{5}-9\right\} = \pi\left\{15+\frac{33}{5}\right\} = \frac{108\pi}{5}$$

b. Let R be the region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$. Find the volume of the solid with base R and cross-sections perpendicular to the x-axis are squares.

Cross-sectional areas: $A(x) = (-x+3-(x^2+1))^2$

$$= (2-x-x^2)^2 = x^4+2x^3-3x^2-4x+4$$

$$\Rightarrow V = \int_{-2}^{1} A(x)\,dx = \frac{1}{5}x^5+\frac{1}{2}x^4-x^3-2x^2+4x\Big|_{-2}^{1}$$

$$= \frac{1}{5}(1+32)+\frac{1}{2}(1-16)-(1+8)-2(1-4)+4(1+2)$$

$$= \frac{33}{5}-\frac{15}{2}-9+6+12$$

$$= \frac{66-75}{10}+9 = -\frac{9}{10}+9 = \frac{81}{10}.$$

13. (10 points) Answer True or False to each of the following and briefly explain your answers.

a. True/False: We have $\int_0^5 |x^2 - 3x - 4|\, dx \geq 0$.

True: $|x^2-3x-4|$ is always ≥ 0, thus so is any definite integral $\int_a^b$ where $b \geq a$.

b. True/False: We have $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \pi^{2k} = -1$.

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \pi^{2k} = \cos \pi$$

Maclaurin series for $\cos x$, evaluated at $x = \pi$.

c. True/False: We have

$$\frac{d}{dx}\left(\int_0^{\pi/4} \cos(x)\, dx\right) = \frac{\sqrt{2}-2}{2}.$$

$= 0$, since $\int_0^{\pi/4} \cos x\, dx$ is a constant.

d. True/False: There is a positive integer m such that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m-1} + \frac{1}{m} > 20$.

Since $\sum_{n=1}^{\infty} \frac{1}{n} = \lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{n} = \infty$,

thus, for any positive N (say $=20$), there is some m such that $\sum_{n=1}^{m} \frac{1}{n} > N$.

14. (10 points) Determine whether each of the following series is convergent or divergent. Indicate test used.

a. $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ convergent : comparison with $\sum \frac{1}{n^2}$.

b. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ convergent : alternating series test.

c. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ convergent : ratio test $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2}{n^2} \cdot \frac{2^n}{2^{n+1}} \xrightarrow[n\to\infty]{} 0 < 1$.

d. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ convergent ; root test $\sqrt[n]{n^2 e^{-n^3}} = (n^{1/n})^2 e^{-n^2} \to 0 < 1$.

e. $\sum_{n=1}^{\infty} \frac{1}{3^n - 1}$ convergent : limit comp test with $\sum \frac{1}{3^n}$

i.e. $\frac{1/3^n}{1/3^n-1} = \frac{3^n-1}{3^n} \xrightarrow[n\to\infty]{} 1$.